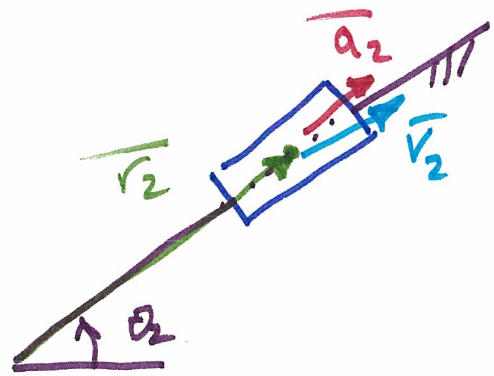
Chapter 7
Acceleration Analysis of Planar Mechanisms

Exam April 7
Ch. 4 & Ch. 6

$$\bar{r}_2 = r_2 e^{j\theta_2}$$

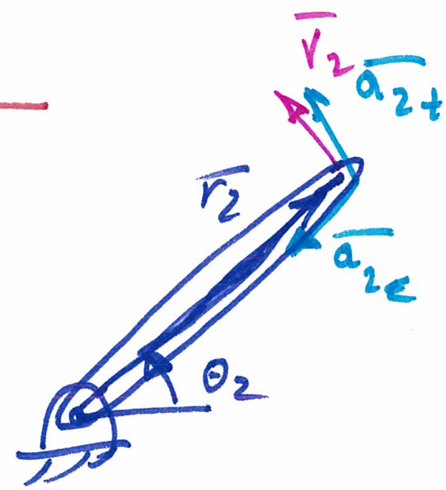
$$\bar{v}_2 = \frac{d}{dt}\left(r_2 e^{j\theta_2}\right) = \dot{r}_2 e^{j\theta_2}$$

$$\bar{a}_2 = \frac{d}{dt}\left(\dot{r}_2 e^{j\theta_2}\right) = \ddot{r}_2 e^{j\theta_2}$$



$$\bar{r}_2 = r_2 e^{j\theta_2}$$

$$\bar{v}_2 = \frac{d}{dt}\left(r_2 e^{j\theta_2}\right) = r_2 j\dot{\theta}_2 e^{j\theta_2}$$



$$\bar{a}_2 = \frac{d}{dt}\left(\bar{v}_2\right) = \frac{d}{dt}\left(r_2 j\dot{\theta}_2 e^{j\theta_2}\right)$$

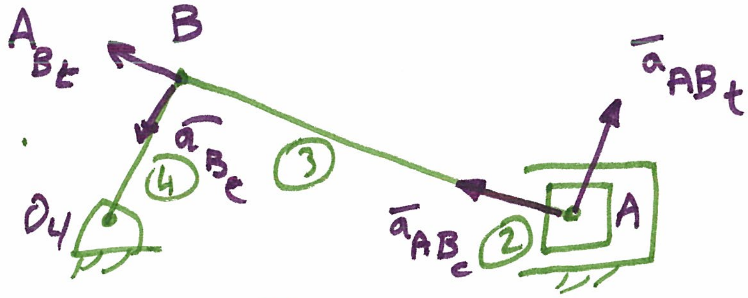
$$= r_2 j \frac{d}{dt}\left(\dot{\theta}_2 e^{j\theta_2}\right)$$

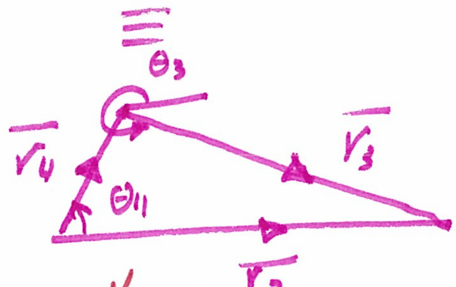
$$= r_2 j\left(\ddot{\theta}_2 e^{j\theta_2} + \dot{\theta}_2 j\dot{\theta}_2 e^{j\theta_2}\right)$$

$$= r_2\ddot{\theta}_2 j e^{j\theta_2} - r_2\dot{\theta}_2^2 e^{j\theta_2}$$

$\bar{a}_{2t} \rightarrow$ tangential $+$ $\bar{a}_{2c} \rightarrow$ centripital

Acceleration Analysis of Internal Combustion Engine





$\theta_2 = 0$

Use the same procedure as in velocity Analysis

$$\overline{r_4} + \overline{r_3} = \overline{r_2}$$

input: r_2
output: θ_3 θ_4

$$r_4 e^{j\theta_4} + r_3 e^{j\theta_3} = r_2 e^{j0} = r_2$$

Differentiate w.r.t. time

$$r_4 \dot{\theta}_4 j e^{j\theta_4} + r_3 \dot{\theta}_3 j e^{j\theta_3} = \dot{r}_2 \longrightarrow$$

solve for $\dot{\theta}_3$ & $\dot{\theta}_4$ in terms of $\dot{r}_2$

Differentiate w.r.t. time

$$r_4 \ddot{\theta}_4 j e^{j\theta_4} - r_4 \dot{\theta}_4^2 e^{j\theta_4} + r_3 \ddot{\theta}_3 j e^{j\theta_3} - r_3 \dot{\theta}_3^2 e^{j\theta_3} = \ddot{r}_2$$

$$\overline{a}_{B_t} + \overline{a}_{B_c} + \overline{a}_{AB_t} + \overline{a}_{AB_c} = \overline{a}_A$$

$$r_4\ddot{\theta}_4 j(\cos\theta_4 + j\sin\theta_4) - r_4\dot{\theta}_4^2(\cos\theta_4 + j\sin\theta_4)$$
$$+ r_3\ddot{\theta}_3 j(\cos\theta_3 + j\sin\theta_3) - r_3\dot{\theta}_3^2(\cos\theta_3 + j\sin\theta_3) = \ddot{r}_2$$

real $\quad -r_4\ddot{\theta}_4\sin\theta_4 - r_4\dot{\theta}_4^2\cos\theta_4 - r_3\ddot{\theta}_3\sin\theta_3 - r_3\dot{\theta}_3^2\cos\theta_3 = \ddot{r}_2$

imag. $\quad r_4\ddot{\theta}_4\cos\theta_4 - r_4\dot{\theta}_4^2\sin\theta_4 + r_3\ddot{\theta}_3\cos\theta_3 - r_3\dot{\theta}_3^2\sin\theta_3 = 0$

Rearranging

$$-r_4\ddot{\theta}_4\sin\theta_4 - r_3\ddot{\theta}_3\sin\theta_3 = \ddot{r}_2 + r_4\dot{\theta}_4^2\cos\theta_4 + r_3\dot{\theta}_3^2\cos\theta_3 = C$$

$$r_4\ddot{\theta}_4\cos\theta_4 + r_3\ddot{\theta}_3\cos\theta_3 = r_4\dot{\theta}_4^2\sin\theta_4 + r_3\dot{\theta}_3^2\sin\theta_3 = D$$

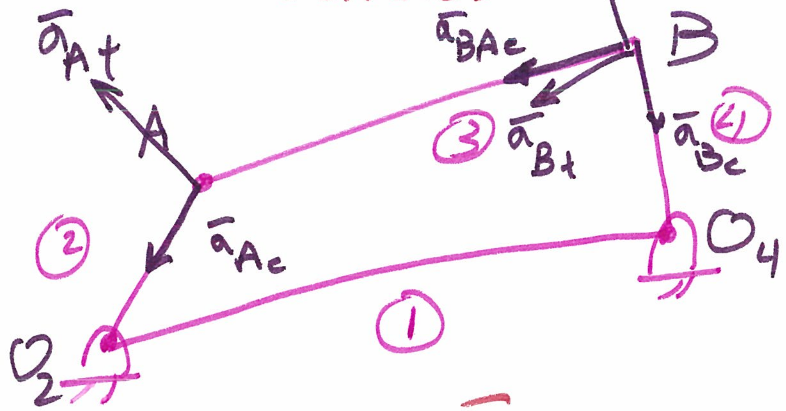
Putting in matrix form

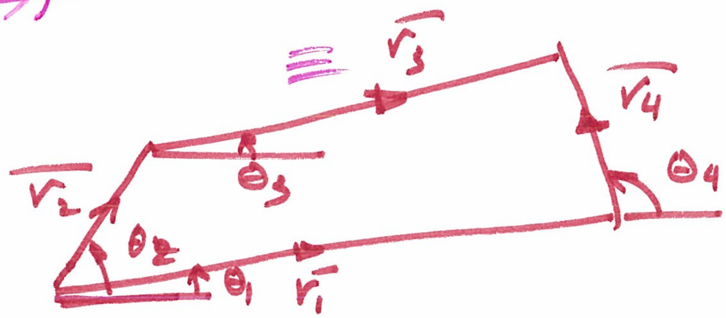
$$\begin{bmatrix} -r_4\sin\theta_4 & -r_3\sin\theta_3 \\ r_4\cos\theta_4 & r_3\cos\theta_3 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_4 \\ \ddot{\theta}_3 \end{Bmatrix} = \begin{Bmatrix} C \\ D \end{Bmatrix}$$

$$\begin{Bmatrix} \ddot{\theta}_4 \\ \ddot{\theta}_3 \end{Bmatrix} = \frac{1}{-r_3 r_4\sin\theta_4\cos\theta_3 + r_3 r_4\sin\theta_3\cos\theta_4} \begin{bmatrix} r_3\cos\theta_3 & r_3\sin\theta_3 \\ -r_4\cos\theta_4 & -r_4\sin\theta_4 \end{bmatrix} \begin{Bmatrix} C \\ D \end{Bmatrix}$$

$$= \frac{1}{r_3 r_4\sin(\theta_3 - \theta_4)} \begin{Bmatrix} r_3 C\cos\theta_3 + r_3 D\sin\theta_3 \\ -r_4 C\cos\theta_4 - r_4 D\sin\theta_4 \end{Bmatrix}$$

Acceleration Analysis of a 4-Bar Mechanism





input: θ_2

output: θ_3 θ_4

$$\bar{r}_2 + \bar{r}_3 = \bar{r}_1 + \bar{r}_4$$

$$r_2 e^{j\theta_2} + r_3 e^{j\theta_3} = r_1 e^{j\theta_1} + r_4 e^{j\theta_4}$$

Differentiate w.r.t. time

$$r_2 \dot{\theta}_2 j e^{j\theta_2} + r_3 \dot{\theta}_3 j e^{j\theta_3} = 0 + r_4 \dot{\theta}_4 j e^{j\theta_4}$$

Differentiate w.r.t. time

$$r_2 \ddot{\theta}_2 j e^{j\theta_2} - r_2 \dot{\theta}_2^2 e^{j\theta_2} + r_3 \ddot{\theta}_3 j e^{j\theta_3} - r_3 \dot{\theta}_3^2 e^{j\theta_3} = r_4 \ddot{\theta}_4 j e^{j\theta_4} - \cancel{r_4 \dot{\theta}_4^2 e} \quad \boxed{-r_4 \dot{\theta}_4^2 e^{j\theta_4}}$$

$$\bar{a}_{At} + \bar{a}_{Ac} + \bar{a}_{BA} + \bar{a}_{BA} = \bar{a}_{Bt} + \bar{a}_{Bc}$$

$$r_2\ddot{\theta}_2 j(\cos\theta_2 + j\sin\theta_2) - r_2\dot{\theta}_2^2(\cos\theta_2 + j\sin\theta_2)$$
$$+ r_3\ddot{\theta}_3 j(\cos\theta_3 + j\sin\theta_3) - r_3\dot{\theta}_3^2(\cos\theta_3 + j\sin\theta_3)$$
$$= r_4\ddot{\theta}_4 j(\cos\theta_4 + j\sin\theta_4) - r_4\dot{\theta}_4^2(\cos\theta_4 + j\sin\theta_4)$$

real $\rightarrow$ $-r_2\ddot{\theta}_2\sin\theta_2 - r_2\dot{\theta}_2^2\cos\theta_2 - r_3\ddot{\theta}_3\sin\theta_3 - r_3\dot{\theta}_2^2\cos\theta_3$
$$= -r_4\ddot{\theta}_4\sin\theta_4 - r_4\dot{\theta}_4^2\cos\theta_4$$

imag. $\rightarrow$ $r_2\ddot{\theta}_2\cos\theta_2 - r_2\dot{\theta}_2^2\sin\theta_2 + r_3\ddot{\theta}_3\cos\theta_3 - r_3\dot{\theta}_2^2\sin\theta_3$
$$= r_4\ddot{\theta}_4\cos\theta_4 - r_4\dot{\theta}_4^2\sin\theta_4$$

Rearranging

$$-r_4\ddot{\theta}_4\sin\theta_4 + r_3\ddot{\theta}_3\sin\theta_3 = -r_2\ddot{\theta}_2\sin\theta_2 - r_2\dot{\theta}_2^2\cos\theta_2 - r_3\dot{\theta}_3^2\cos\theta_3 + r_4\dot{\theta}_4^2\cos\theta_4 = C$$

$$r_4\ddot{\theta}_4\cos\theta_4 - r_3\ddot{\theta}_3\cos\theta_3 = r_2\ddot{\theta}_2\cos\theta_2 - r_2\dot{\theta}_2^2\sin\theta_2 - r_3\dot{\theta}_3^2\sin\theta_3 + r_4\dot{\theta}_4^2\sin\theta_4 = D$$

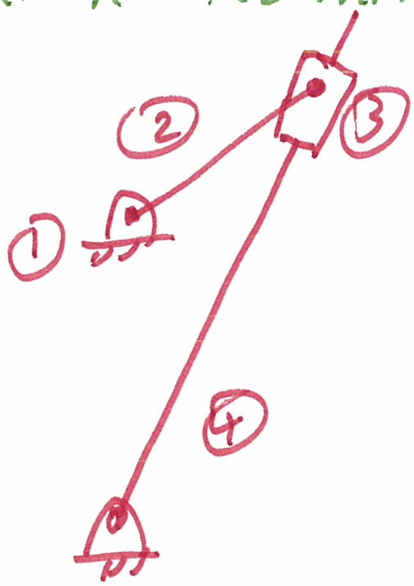
Putting in matrix form

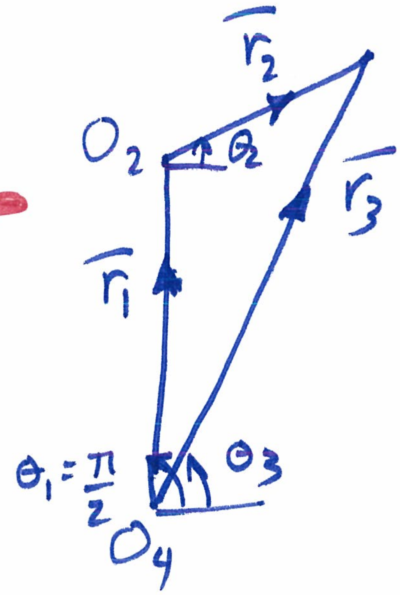
$$\begin{bmatrix} -r_4\sin\theta_4 & r_3\sin\theta_3 \\ r_4\cos\theta_4 & -r_3\cos\theta_3 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_4 \\ \ddot{\theta}_3 \end{Bmatrix} = \begin{Bmatrix} C \\ D \end{Bmatrix}$$

$$\begin{Bmatrix} \ddot{\theta}_4 \\ \ddot{\theta}_3 \end{Bmatrix} = \frac{1}{r_3 r_4\sin\theta_4\cos\theta_3 - r_3 r_4\sin\theta_3\cos\theta_4} \begin{bmatrix} -r_3\cos\theta_3 & -r_3\sin\theta_3 \\ -r_4\cos\theta_4 & -r_4\sin\theta_4 \end{bmatrix} \begin{Bmatrix} C \\ D \end{Bmatrix}$$

$$\begin{Bmatrix} \ddot{\theta}_4 \\ \\ \ddot{\theta}_3 \end{Bmatrix} = \frac{1}{r_3 r_4 \sin(\theta_4 - \theta_3)} \begin{Bmatrix} -r_3 C \cos\theta_3 - r_3 D \sin\theta_3 \\ \\ -r_4 C \cos\theta_4 - r_4 D \sin\theta_4 \end{Bmatrix}$$

Acceleration Analysis of the RRPR Quick-Return Mechanism





A: on links 2 & 3

B: on link 4

input: θ_2

output: θ_3 r_3

$$\overline{r_1} + \overline{r_2} = \overline{r_3}$$

$$r_1 e^{j\frac{\pi}{2}} + r_2 e^{j\theta_2} = r_3 e^{j\theta_3}$$

Differentiate w.r.t. time

$$0 + r_2 \dot{\theta}_2 j e^{j\theta_2} = \dot{r}_3 e^{j\theta_3} + r_3 \dot{\theta}_3 j e^{j\theta_3} \rightarrow$$

solve for $\dot{\theta}_3$ $\dot{r}_3$ in terms of $\dot{\theta}_2$

Differentiate w.r.t. time

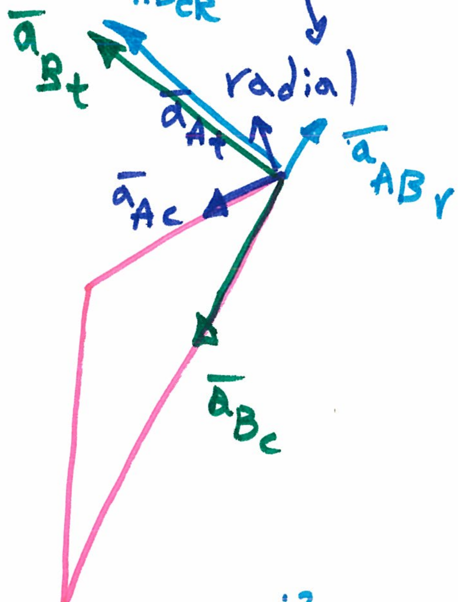
result of 1st term

$$r_2 \ddot{\theta}_2 j e^{j\theta_2} - r_2 \dot{\theta}_2^2 e^{j\theta_2} = \ddot{r}_3 e^{j\theta_3} + \dot{r}_3 \dot{\theta}_3 j e^{j\theta_3}$$

2nd term $\rightarrow$ $$+ \dot{r}_3 \dot{\theta}_3 j e^{j\theta_3} + r_3 \ddot{\theta}_3 j e^{j\theta_3} - r_3 \dot{\theta}_3^2 e^{j\theta_3}$$

$$r_2 \ddot{\theta}_2 j e^{j\theta_2} - r_2 \dot{\theta}_2^2 e^{j\theta_2} = \ddot{r}_3 e^{j\theta_3} + 2 \dot{r}_3 \dot{\theta}_3 j e^{j\theta_3} + r_3 \ddot{\theta}_3 j e^{j\theta_3} - r_3 \dot{\theta}_3^2 e^{j\theta_3}$$

$$\bar{a}_{A_t} + \bar{a}_{A_c} = \underbrace{\bar{a}_{AB_r}}_{\text{radial}} + \underbrace{\bar{a}_{AB_{CR}}}_{\text{Coriolis}} + \bar{a}_{B_t} + \bar{a}_{B_c}$$



$$r_2 \ddot{\theta}_2 j(\cos\theta_2 + j\sin\theta_2) - r_2 \dot{\theta}_2^2 (\cos\theta_2 + j\sin\theta_2)$$
$$= \ddot{r}_3 (\cos\theta_3 + j\sin\theta_3) + 2\dot{r}_3 \dot{\theta}_3 j(\cos\theta_3 + j\sin\theta_3)$$
$$+ r_3 \ddot{\theta}_3 j(\cos\theta_3 + j\sin\theta_3) - r_3 \dot{\theta}_3^2 (\cos\theta_3 + j\sin\theta_3)$$

real $\rightarrow$ $-r_2\ddot{\theta}_2\sin\theta_2 - r_2\dot{\theta}_2^2\cos\theta_2$

& $= \ddot{r}_3\cos\theta_3 - 2\dot{r}_3\dot{\theta}_3\sin\theta_3 - r_3\ddot{\theta}_3\sin\theta_3 - r_3\dot{\theta}_3^2\cos\theta_3$

imag $\rightarrow$ $r_2\ddot{\theta}_2\cos\theta_2 - r_2\dot{\theta}_2^2\sin\theta_2$

$= \ddot{r}_3\sin\theta_3 + 2\dot{r}_3\dot{\theta}_3\cos\theta_3 + r_3\ddot{\theta}_3\cos\theta_3 - r_3\dot{\theta}_3^2\sin\theta_3$

Rearranging to have $\ddot{r}_3$ & $\ddot{\theta}_3$ on the same side alone

$$\ddot{r}_3\cos\theta_3 - r_3\ddot{\theta}_3\sin\theta_3 = -r_2\ddot{\theta}_2\sin\theta_2 - r_2\dot{\theta}_2^2\cos\theta_2 + 2\dot{r}_3\dot{\theta}_3\sin\theta_3 + r_3\dot{\theta}_3^2\cos\theta_3 = C$$

$$\ddot{r}_3\sin\theta_3 + r_3\ddot{\theta}_3\cos\theta_3 = r_2\ddot{\theta}_2\cos\theta_2 - r_2\dot{\theta}_2^2\sin\theta_2 - 2\dot{r}_3\dot{\theta}_3\cos\theta_3 + r_3\dot{\theta}_3^2\sin\theta_3 = D$$

Move to a matrix form

$$\begin{bmatrix} \cos\theta_3 & -r_3\sin\theta_3 \\ \sin\theta_3 & r_3\cos\theta_3 \end{bmatrix} \begin{Bmatrix} \ddot{r}_3 \\ \ddot{\theta}_3 \end{Bmatrix} = \begin{Bmatrix} C \\ D \end{Bmatrix}$$

$$\begin{Bmatrix} \ddot{r}_3 \\ \ddot{\theta}_3 \end{Bmatrix} = \frac{1}{r_3\cos^2\theta_3 + r_3\sin^2\theta_3} \begin{bmatrix} r_3\cos\theta_3 & r_3\sin\theta_3 \\ -\sin\theta_3 & \cos\theta_3 \end{bmatrix} \begin{Bmatrix} C \\ D \end{Bmatrix}$$

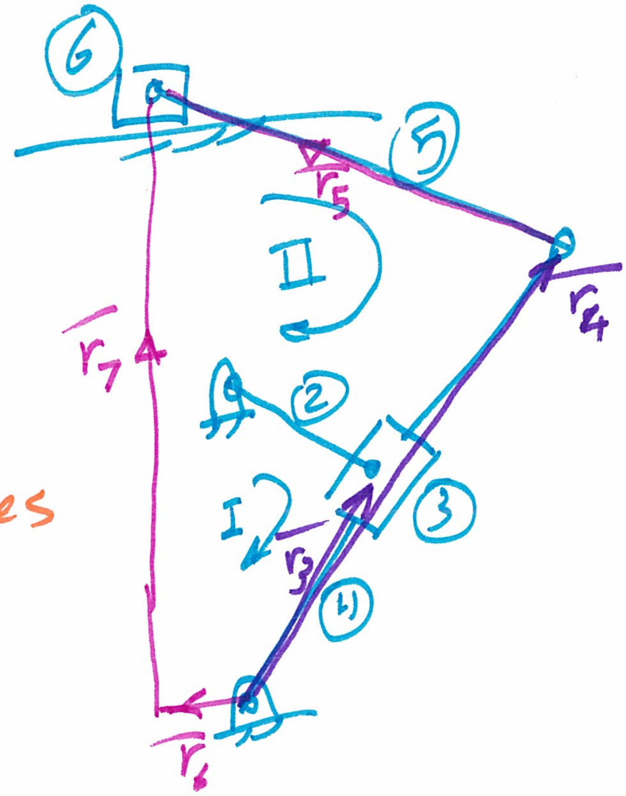
$$= \frac{1}{r_3}\begin{Bmatrix} r_3 C\cos\theta_3 + r_3 D\sin\theta_3 \\ -C\sin\theta_3 + D\cos\theta_3 \end{Bmatrix}$$

Acceleration Analysis of Multi-Loop Machines

Shaper QRM

Loop I: 1 2 3 4

Loop II: 1 4 5 6



Loops are in series

Loop I:

input: θ_2

output: r_3 $\theta_3 = \theta_4$

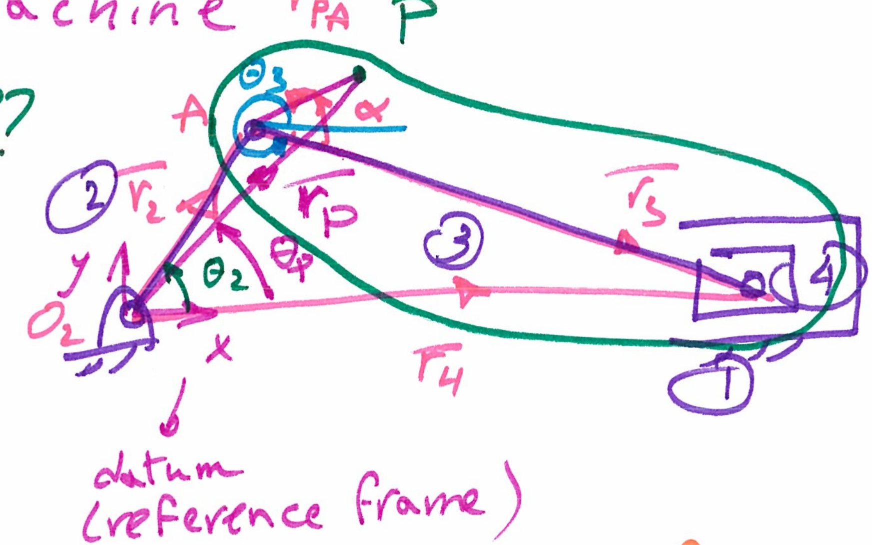
Loop II:

input: θ_4

output: θ_5 r_6

Acceleration Analysis of a Point that Is $\boxed{\text{NOT}}$ a Part of the Vector Loop of the Machine

Acceleration of P?



* Create vector loop equation for the machine

$$\overline{r_2} + \overline{r_3} = \overline{r_4} \longrightarrow \text{solve for } \theta_3 \text{ \& } r_4 \text{ in terms of } \theta_2$$

* Differentiate twice to find velocity & acceleration unknowns:

$$\dot{\theta}_3 \quad \dot{r}_4 \quad \ddot{\theta}_3 \quad \ddot{r}_4$$

* Create vector loop equation for point P

$$\overline{r_2} + \overline{r_{PA}} = \overline{r_P}$$

$$r_2 e^{j\theta_2} + r_{PA} e^{j(\theta_3 + \alpha)} = r_P e^{j\theta_P}$$

Differentiate w.r.t. time

$$r_2\dot{\theta}_2 j e^{j\theta_2} + r_{PA}\dot{\theta}_3 j e^{j(\theta_3+\alpha)} = \dot{r}_P e^{j\theta_P} + r_P\dot{\theta}_P j e^{j\theta_P}$$

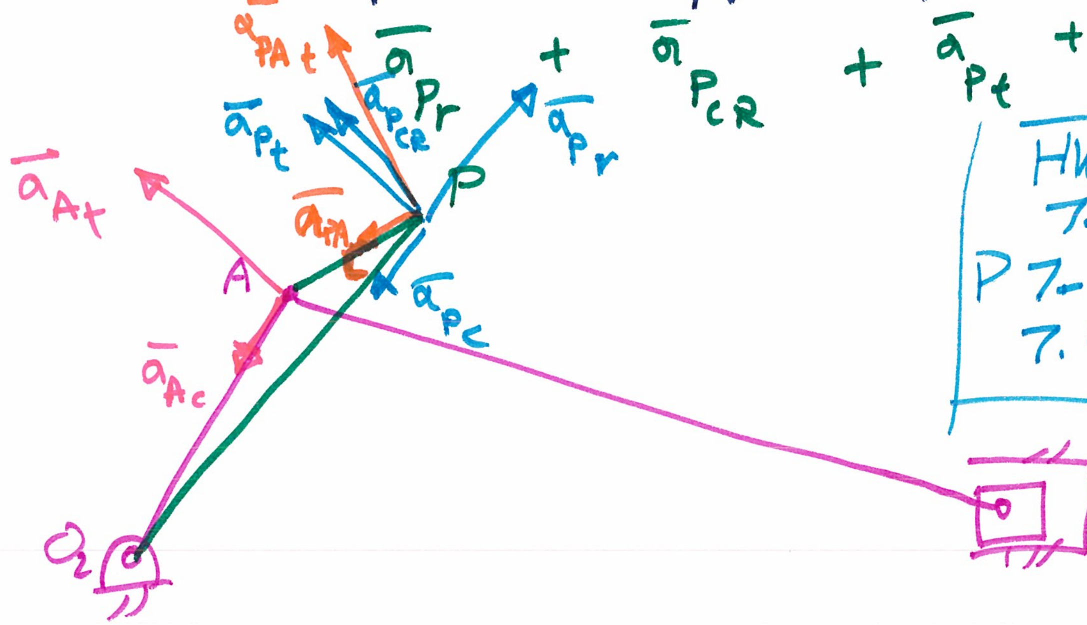
↳ solve for $\dot{r}_P$ & $\dot{\theta}_P$

Differentiate w.r.t. time

$$\underbrace{r_2\ddot{\theta}_2 j e^{j\theta_2}}_{\bar{a}_{At}} \underbrace{- r_2\dot{\theta}_2^2 e^{j\theta_2}}_{+\ \bar{a}_{Ac}} \underbrace{+ r_{PA}\ddot{\theta}_3 j e^{j(\theta_3+\alpha)}}_{+\ \bar{a}_{PAt}} \underbrace{- r_{PA}\dot{\theta}_3^2 e^{j(\theta_3+\alpha)}}_{+\ \bar{a}_{PAc}}$$

$$= \ddot{r}_P e^{j\theta_P} + \dot{r}_P\dot{\theta}_P j e^{j\theta_P} + \dot{r}_P\dot{\theta}_P j e^{j\theta_P} + r_P\ddot{\theta}_P j e^{j\theta_P} - r_P\dot{\theta}_P^2 e^{j\theta_P}$$

$$= \underbrace{\ddot{r}_P e^{j\theta_P}}_{\bar{a}_{Pr}} + \underbrace{2\dot{r}\dot{\theta}_P j e^{j\theta_P}}_{\bar{a}_{PcR}} + \underbrace{r_P\ddot{\theta}_P j e^{j\theta_P}}_{\bar{a}_{Pt}} + \underbrace{- r_P\dot{\theta}_P^2 e^{j\theta_P}}_{\bar{a}_{Pc}}$$



HW
7.8
P 7-6 (a)
7.4

Solve the equations using complex phasors to find expressions for the unknowns: $\ddot{r}_P$ $\ddot{\theta}_P$