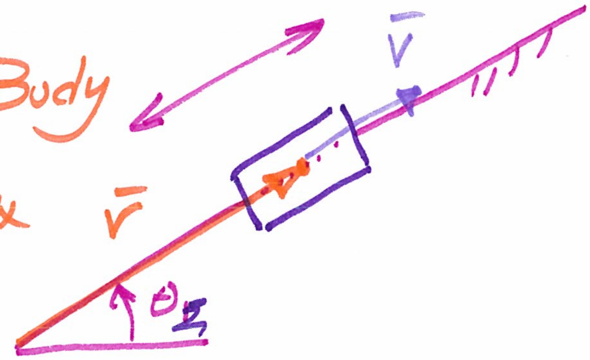
Chapter 6
Velocity Analysis of Planar Machines

$$\bar{V} = \frac{d\bar{r}}{dt}$$

Example: Translating Body

$\bar{r} = r e^{j\theta}$ (in complex phasor notation)



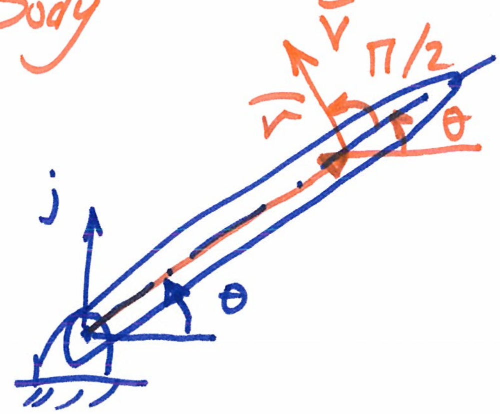
$$\bar{V} = \frac{d\bar{r}}{dt}$$

$$= \frac{d}{dt}\left(r e^{j\theta}\right)$$

$$= e^{j\theta} \frac{d}{dt}(r)$$

$$= \dot{r} e^{j\theta}$$

Example: Rotating Body



$$\bar{r} = r e^{j\theta}$$

$$\bar{V} = \frac{d\bar{r}}{dt}$$

$$= \frac{d}{dt}\left(r e^{j\theta}\right)$$

$$= r \frac{d}{dt}\left(e^{j\theta}\right)$$

$$= r\left(e^{j\theta}\frac{d}{dt}(j\theta)\right)$$

$$= r\left(e^{j\theta}\, j\frac{d\theta}{dt}\right)$$

$$= r\left(e^{j\theta}\, j\,\dot{\theta}\right)$$

$$= (r\dot{\theta})\, j\, e^{j\theta}$$

$$= (r\dot{\theta})\underbrace{\left(e^{j\frac{\pi}{2}}\right)}_{\text{along imaginary axis}}\left(e^{j\theta}\right) = \underbrace{(r\dot{\theta})}_{\text{magnitude}}\, e^{j\overbrace{\left(\theta+\frac{\pi}{2}\right)}^{\text{direction}}}$$

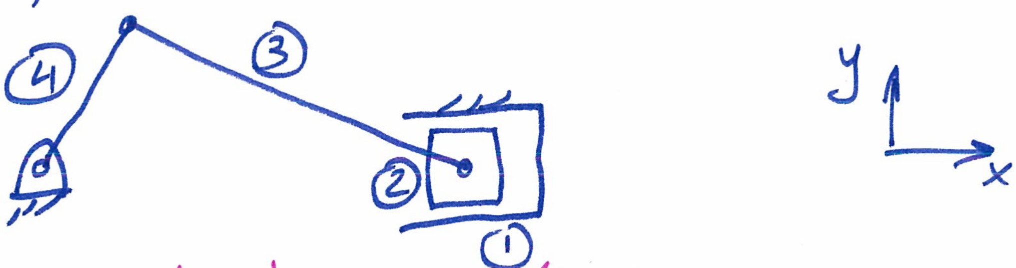
Reminder

$$\frac{d}{dx}\left(e^{x^2}\right)$$

$$= e^{x^2}\left(\frac{d}{dx}x^2\right)$$

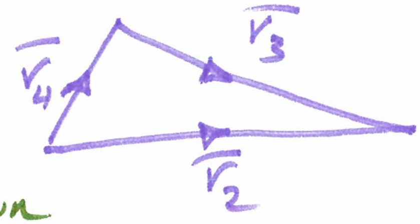
$$= 2x\, e^{x^2}$$

Example: Internal-Combustion Engine



Velocity Analysis Steps:

① Create equivalent vector loop



② Create vector loop equation

$$\bar{r}_2 = \bar{r}_4 + \bar{r}_3$$

③ Analyze displacement
(input: r_2
output: θ_3 θ_4)

④ Express vector loop equation in terms of complex phasors

$$r_2 = r_4 e^{j\theta_4} + r_3 e^{j\theta}$$

⑤ Differentiate with respect to time

$$\dot{r}_2 = r_4 \dot{\theta}_4 j e^{j\theta_4} + r_3 \dot{\theta}_3 j e^{j\theta_3}$$

⑥ Replace $e^{j\theta}$ by $(\cos\theta + j\sin\theta)$

$$\dot{r}_2 = r_4 \dot{\theta}_4 j(\cos\theta_4 + j\sin\theta_4$$
$$+ r_3\dot{\theta}_3 j(\cos\theta_3 + j\sin\theta$$

⑦ Separate into real & imaginary equations

real → $\dot{r}_2 = -r_4\dot{\theta}_4\sin\theta_4 - r_3\dot{\theta}_3 \sin$
imag. → $0 = r_4\dot{\theta}_4\cos\theta_4 + r_3\dot{\theta}_3\cos$

⑧ Solve for the unknowns ($\dot{\theta}_3$ & $\dot{\theta}_4$) in terms of the input, $\dot{r}_2$

~~Wewr~~
Rewrite the equations in a matrix form

$$\begin{Bmatrix}\dot{r}_2\\ 0\end{Bmatrix} = \begin{bmatrix}-r_4\sin\theta_4 & -r_3\sin\theta_3\\ r_4\cos\theta_4 & r_3\cos\theta_3\end{bmatrix}\begin{Bmatrix}\dot{\theta}_4\\ \dot{\theta}_3\end{Bmatrix}$$

$$\{C\} = [A] \qquad \{B\}$$

$$\begin{Bmatrix}\dot{\theta}_4\\ \dot{\theta}_3\end{Bmatrix} = \begin{bmatrix}-r_4\sin\theta_4 & -r_3\sin\theta_3\\ r_4\cos\theta_4 & r_3\cos\theta_3\end{bmatrix}^{-1}\begin{Bmatrix}\dot{r}_2\\ 0\end{Bmatrix}$$

$$\begin{Bmatrix}\dot{\theta}_4\\ \dot{\theta}_3\end{Bmatrix} = \frac{1}{-r_4 r_3\sin\theta_4\cos\theta_3 + r_4 r_3\sin\theta_3\cos\theta_4}\begin{bmatrix}r_3\cos\theta_3 & r_3\sin\theta_3\\ -r_4\cos\theta_4 & -r_4\sin\theta_4\end{bmatrix}\begin{Bmatrix}\dot{r}_2\\ 0\end{Bmatrix}$$

$$= \frac{1}{r_4 r_3\sin(\theta_3-\theta_4)}\begin{Bmatrix}\dot{r}_2 r_3\cos\theta_3\\ -\dot{r}_2 r_4\cos\theta_4\end{Bmatrix}$$

$$= \begin{Bmatrix}\dfrac{\dot{r}_2\cos\theta_3}{r_4\sin(\theta_3-\theta_4)}\\[2ex] \dfrac{-\dot{r}_2\cos\theta_4}{r_3\sin(\theta_3-\theta_4)}\end{Bmatrix}$$

Reminder

$$\{C\}=[A]\{B\}$$

$n\times 1$ column; coeff. $n\times n$; unknown $n\times 1$

Example:

$a+b=5$

$a-2b=3$

$$\begin{Bmatrix}5\\3\end{Bmatrix}=\begin{bmatrix}1 & 1\\ 1 & -2\end{bmatrix}\begin{Bmatrix}a\\b\end{Bmatrix}$$

Pre-multiply

$$[A]^{-1}\{C\}=[A]^{-1}[A]\{B\}$$

$$[A]^{-1}\{C\}=\underbrace{[I]}_{\text{identity matrix}}\{B\}$$

$$\{B\}=[A]^{-1}\{C\}$$

2x2 Matrix inversion

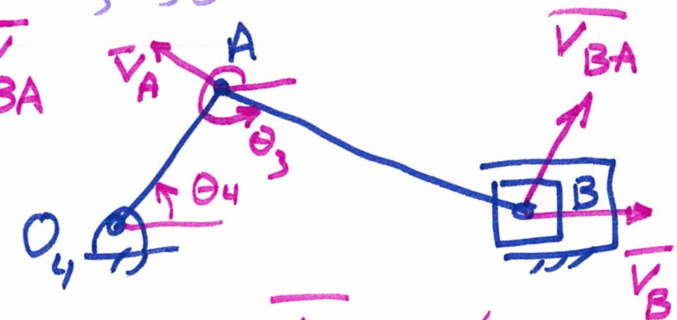
$$[A]=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$$

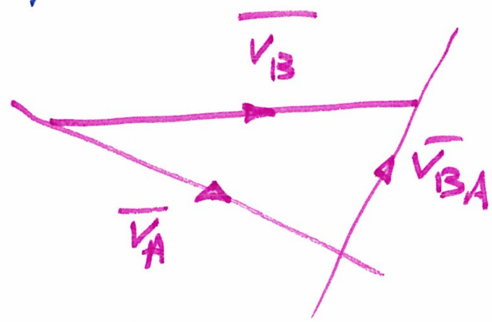
$$[A]^{-1}=\frac{1}{\underbrace{|A|}_{ad-bc}}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$$

9 Interpretation of the velocity terms
(Could be ~~e~~earlier after 5)

$$\dot r_2 = r_4 \dot\theta_4 j e^{j\theta_4} + r_3 \dot\theta_3 j e^{j\theta_3}$$

$$\overline{V_B} = \overline{V_A} + \overline{V_{BA}}$$





Check results

$$\begin{Bmatrix} \dot\theta_4 \\ \dot\theta_3 \end{Bmatrix} = \begin{Bmatrix} \dfrac{\dot r_2 \cos\theta_3}{r_4 \sin(\theta_3-\theta_4)} \\ \dfrac{-\dot r_2 \cos\theta_4}{r_3 \sin(\theta_3-\theta_4)} \end{Bmatrix}$$

$\dot\theta_3 = \dot\theta_4 = \infty$ when $\sin(\theta_3-\theta_4)=0$

$(\theta_3-\theta_4)=0$ or π

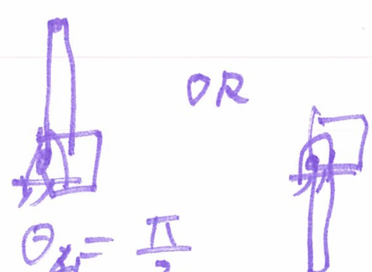
$\theta_3=\theta_4$ or $\theta_3=\theta_4+\pi$

$\dot\theta_3 \cancel{= \dot\theta_4} = 0$ when $\cos\theta_4 = 0$

$\theta_4 = \frac{\pi}{2}$ or $\frac{3\pi}{2}$

$\dot\theta_4 = 0$ when $\cos\theta_3 = 0$

$\theta_3 = \frac{\pi}{2}$ or $\frac{3\pi}{2}$



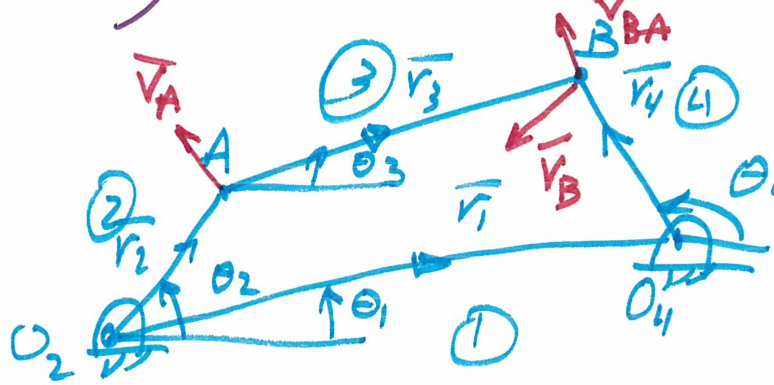
$\theta_4 = \frac{\pi}{2}$

$\theta_3 = \frac{3\pi}{2}$

$\boxed{r_3 = r_4}$

Velocity Analysis of 4-Bar Mechanism

1



2 $\quad \bar{r}_2 + \bar{r}_3 = \bar{r}_1 + \bar{r}_4$

3 $\quad r_2 e^{j\theta_2} + r_3 e^{j\theta_3} = r_1 e^{j\theta_1} + r_4 e^{j\theta_4} \rightarrow$ Analyze disp.

4 $\quad$ input: θ_2 $\qquad$ output: θ_3 & θ_4

5 $\quad$ Diff. w.r.t. time

$$\underbrace{r_2 \dot{\theta}_2 j e^{j\theta_2}}_{\bar{V}_A} + \underbrace{r_3 \dot{\theta}_3 j e^{j\theta_3}}_{\bar{V}_{BA}} = 0 + \underbrace{r_4 \dot{\theta}_4 j e^{j\theta_4}}_{\bar{V}_B}$$

6 $\quad r_2 \dot{\theta}_2 j (\cos\theta_2 + j\sin\theta_2) + r_3 \dot{\theta}_3 j (\cos\theta_3 + j\sin\theta_3)$

$$= r_4 \dot{\theta}_4 j (\cos\theta_4 + j\sin\theta_4)$$

7 $\quad$ real $\rightarrow$ $-r_2\dot{\theta}_2 \sin\theta_2 - r_3 \dot{\theta}_3 \sin\theta_3 = -r_4 \dot{\theta}_4 \sin\theta_4$

imag. $\rightarrow$ $r_2\dot{\theta}_2 \cos\theta_2 + r_3 \dot{\theta}_3 \cos\theta_3 = r_4 \dot{\theta}_4 \cos\theta_4$

8 $\quad$ solve for the unknowns: $\dot{\theta}_3$ & $\dot{\theta}_4$

Rearranging to have unknowns in the same side

$$-r_2\dot{\theta}_2\sin\theta_2 = r_3\dot{\theta}_3\sin\theta_3 - r_4\dot{\theta}_4\sin\theta_4$$

$$r_2\dot{\theta}_2\cos\theta_2 = -r_3\dot{\theta}_3\cos\theta_3 + r_4\dot{\theta}_4\cos\theta_4$$

$$\begin{bmatrix} r_3\sin\theta_3 & -r_4\sin\theta_4 \\ -r_3\cos\theta_3 & r_4\cos\theta_4 \end{bmatrix} \begin{Bmatrix} \dot{\theta}_3 \\ \dot{\theta}_4 \end{Bmatrix} = \begin{Bmatrix} -r_2\dot{\theta}_2\sin\theta_2 \\ r_2\dot{\theta}_2\cos\theta_2 \end{Bmatrix}$$

$$[A]\{B\} = \{C\}$$

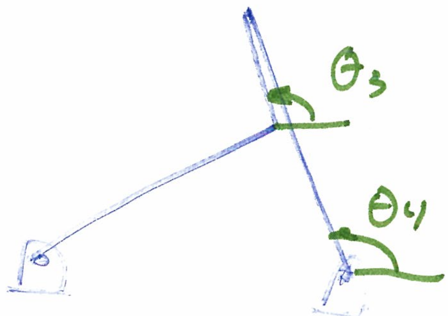
$$\{B\} = [A]^{-1}\{C\}$$

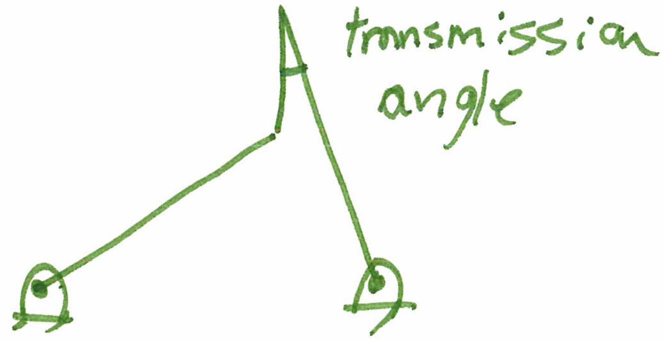
$$\begin{Bmatrix} \dot{\theta}_3 \\ \dot{\theta}_4 \end{Bmatrix} = \frac{1}{r_3r_4\sin\theta_3\cos\theta_4 - r_3r_4\sin\theta_4\cos\theta_3} \begin{bmatrix} r_4\cos\theta_4 & r_4\sin\theta_4 \\ r_3\cos\theta_3 & r_3\sin\theta_3 \end{bmatrix} \begin{Bmatrix} -r_2\dot{\theta}_2\sin\theta_2 \\ r_2\dot{\theta}_2\cos\theta_2 \end{Bmatrix}$$

$$= \frac{r_2\dot{\theta}_2}{r_3r_4\sin(\theta_3-\theta_4)} \begin{Bmatrix} -r_4\cos\theta_4\sin\theta_2 + r_4\sin\theta_4\cos\theta_2 \\ -r_3\cos\theta_3\sin\theta_2 + r_3\sin\theta_3\cos\theta_2 \end{Bmatrix}$$

$$= \frac{r_2\dot{\theta}_2}{r_3r_4\sin(\theta_3-\theta_4)} \begin{Bmatrix} r_4\sin(\theta_4-\theta_2) \\ r_3\sin(\theta_3-\theta_2) \end{Bmatrix}$$

$$\begin{Bmatrix} \dot{\theta}_3 \\ \dot{\theta}_4 \end{Bmatrix} = r_2\dot{\theta}_2 \begin{Bmatrix} \dfrac{\sin(\theta_4-\theta_2)}{r_3\sin(\theta_3-\theta_4)} \\ \dfrac{\sin(\theta_3-\theta_2)}{r_4\sin(\theta_3-\theta_4)} \end{Bmatrix}$$



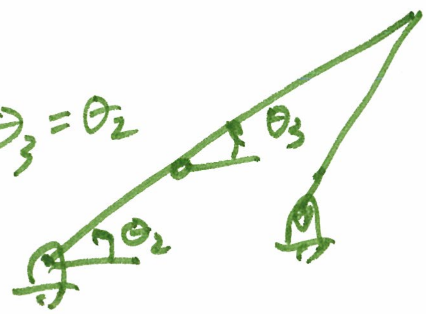


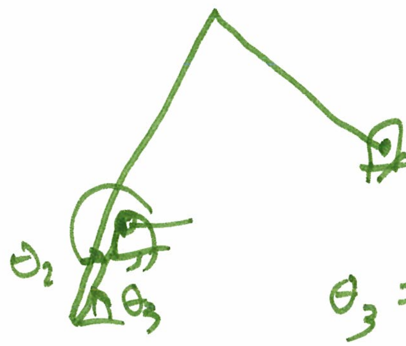
$$\theta_4 = \theta_3$$

$$\theta_4 - \theta_3 = 0$$

$$\sin(\theta_3 - \theta_4) = 0 \longrightarrow \dot{\theta}_3 = \dot{\theta}_4 = \infty$$

$\theta_3 = \theta_2$





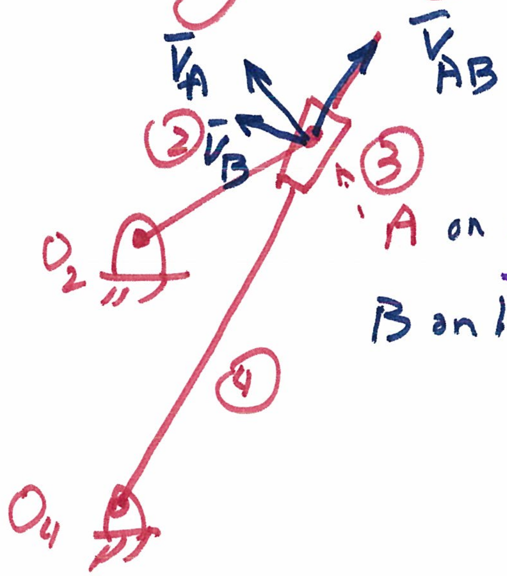
$\dot{\theta}_4 = 0$

ERP

$\theta_3 = \theta_2 - \pi$

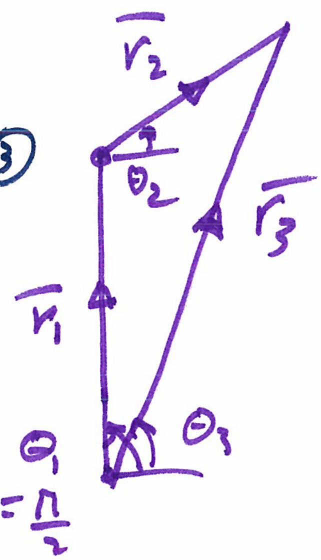
ELP $\dot{\theta}_4 = 0$

Velocity Analysis of RRPR Mechanism



A on links 2 & 3 $\longrightarrow$ B on link 4

1



2 $\bar r_1 + \bar r_2 = \bar r_3 \longrightarrow$ ~~solve~~

$$\frac{d}{dx}(ab) = \frac{da}{dx}b + a\frac{db}{dx}$$

3 input: θ_2

output: r_3 θ_3

4 $r_1 e^{j\theta_1} + r_2 e^{j\theta_2} = r_3 e^{j\theta_3}$

5 Differentiate w.r.t. time

$$0 + r_2 \dot\theta_2 j e^{j\theta_2} = \dot r_3 e^{j\theta_3} + r_3 \dot\theta_3 j e^{j\theta_3}$$

$$\bar V_A = \bar V_{AB} + \bar V_B$$

(6) $r_2 \dot\theta_2 j(\cos\theta_2 + j\sin\theta_2)$
$= \dot r_3(\cos\theta_3 + j\sin\theta_3)$
$+ r_3\dot\theta_3 j(\cos\theta_3 + j\sin\theta_3)$

(7) real $\rightarrow$ $-r_2\dot\theta_2\sin\theta_2 = \dot r_3\cos\theta_3 - r_3\dot\theta_3\sin\theta_3$

imag. $\rightarrow$ $r_2\dot\theta_2\cos\theta_2 = \dot r_3\sin\theta_3 + r_3\dot\theta_3\cos\theta_3$

(8)
$$\begin{bmatrix}\cos\theta_3 & -r_3\sin\theta_3\\ \sin\theta_3 & r_3\cos\theta_3\end{bmatrix}\begin{Bmatrix}\dot r_3\\ \dot\theta_3\end{Bmatrix} = \begin{Bmatrix}-r_2\dot\theta_2\sin\theta_2\\ r_2\dot\theta_2\cos\theta_2\end{Bmatrix}$$

$$\begin{Bmatrix}\dot r_3\\ \dot\theta_3\end{Bmatrix} = \frac{r_2\dot\theta_2}{r_3\cos^2\theta_3 + r_3\sin^2\theta_3}\begin{bmatrix}r_3\cos\theta_3 & r_3\sin\theta_3\\ -\sin\theta_3 & \cos\theta_3\end{bmatrix}\begin{Bmatrix}-\sin\theta_2\\ \cos\theta_2\end{Bmatrix}$$

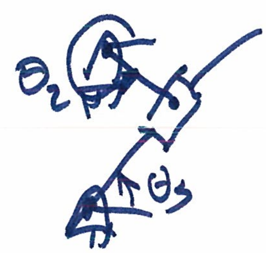
HW 6.9
Fig. P6.7(c)
~~Fig. P6.6(a)~~
Fig. P6-6(a)

$$= \frac{r_2\dot\theta_2}{r_3}\begin{Bmatrix}-r_3\sin\theta_2\cos\theta_3 + r_3\sin\theta_3\cos\theta_2\\ \sin\theta_2\sin\theta_3 + \cos\theta_3\cos\theta_2\end{Bmatrix}$$

$$= \frac{r_2\dot\theta_2}{r_3}\begin{Bmatrix}r_3\sin(\theta_3-\theta_2)\\ \cos(\theta_3-\theta_2)\end{Bmatrix}$$

$\dot\theta_3 = 0 \rightarrow \cos(\theta_3-\theta_2) = 0$

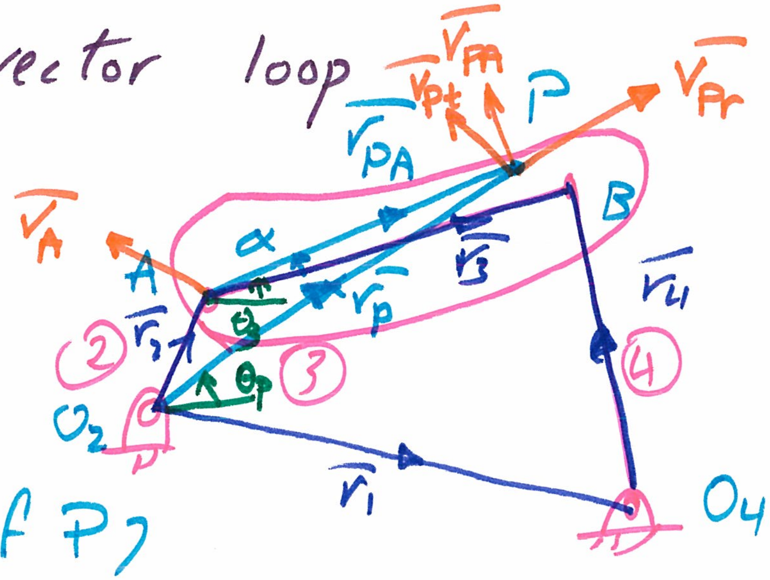
$\theta_3 - \theta_2 = \frac{\pi}{2}$ or $\frac{3\pi}{2}$



$\dot r_3 = 0 \rightarrow \sin(\theta_3-\theta_2) = 0$

$\theta_3 - \theta_2 = 0$ or π

Velocity Analysis for a Point <u>not</u> on the vector loop



Velocity of P?

α: fixed

$\bar{r}_{PA}$: relative position of P w.r.t. A

$\bar{r}_p$: absolute position

Analyze displacement & velocity loops

$$\bar{r}_2 + \bar{r}_3 = \bar{r}_1 + \bar{r}_4$$

$$r_2 e^{j\theta_2} + r_3 e^{j\theta_3} = r_1 e^{j\theta_1} + r_4 e^{j\theta_4} \longrightarrow \text{solve for } \theta_3 \,\&\, \theta_4 \text{ in terms of } \theta_2$$

Differentiate w.r.t. time

$$r_2 \dot{\theta}_2 j e^{j\theta_2} + r_3 \dot{\theta}_3 j e^{j\theta_3} = 0 + r_4 \dot{\theta}_4 j e^{j\theta_4} \longrightarrow \text{" "} \ \dot{\theta}_3 \,\&\, \dot{\theta}_4 \text{ " " " } \dot{\theta}_2$$

$$\bar{r}_2 + \bar{r}_{PA} = \bar{r}_p$$

$$r_2 e^{j\theta_2} + r_{PA} e^{j(\theta_3 + \alpha)} = r_p e^{j\theta_p} \longrightarrow \text{solve for } r_p \,\&\, \theta_p$$

Differentiate w.r.t. time

$$r_2 \dot{\theta}_2 j e^{j\theta_2} + r_{PA} (\dot{\theta}_3 + 0) j e^{j(\theta_3+\alpha)} = \dot{r}_p e^{j\theta_p} + r_p \dot{\theta}_p j e^{j\theta_p}$$

$$\overline{V}_A + \overline{V}_{PA} = \overline{V}_{P_r} + \overline{V}_{P_t}$$

0 because α is fixed

$\overline{V}_{P_r}$: radial; $\overline{V}_{P_t}$: tangential

$$r_2 \dot{\theta}_2 j (\cos\theta_2 + j\sin\theta_2) + r_{PA} \dot{\theta}_3 j (\cos(\theta_3+\alpha) + j\sin(\theta_3+\alpha))$$

$$= \dot{r}_p (\cos\theta_p + j\sin\theta_p) + r_p \dot{\theta}_p j (\cos\theta_p + j\sin\theta_p)$$

real $\rightarrow$ $\underbrace{-r_2 \dot{\theta}_2 \sin\theta_2 - r_{PA} \dot{\theta}_3 \sin(\theta_3+\alpha)}_{A} = \dot{r}_p \cos\theta_p - r_p \dot{\theta}_p \sin\theta_p$

imag. $\rightarrow$ $\underbrace{r_2 \dot{\theta}_2 \cos\theta_2 + r_{PA} \dot{\theta}_3 \cos(\theta_3+\alpha)}_{B} = \dot{r}_p \sin\theta_p + r_p \dot{\theta}_p \cos\theta_p$

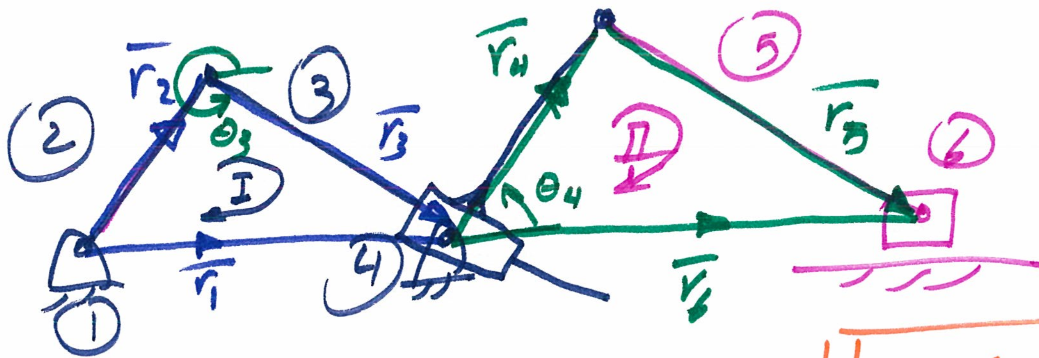
Expressing in matrix form

$$\begin{bmatrix} \cos\theta_p & -r_p\sin\theta_p \\ \sin\theta_p & r_p\cos\theta_p \end{bmatrix} \begin{Bmatrix} \dot{r}_p \\ \dot{\theta}_p \end{Bmatrix} = \begin{Bmatrix} A \\ B \end{Bmatrix}$$

$$\begin{Bmatrix} \dot{r}_p \\ \dot{\theta}_p \end{Bmatrix} = \frac{1}{r_p\cos^2\theta_p + r_p\sin^2\theta_p} \begin{bmatrix} r_p\cos\theta_p & r_p\sin\theta_p \\ -\sin\theta_p & \cos\theta \end{bmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix}$$

$$= \frac{1}{r_p} \begin{Bmatrix} r_p A\cos\theta_p + r_p B\sin\theta_p \\ -A\sin\theta_p + B\cos\theta_p \end{Bmatrix}$$

Velocity Analysis of a Multi-Loop Mechanism



Loops are in series

Loop I: (1) (2) (3) (4)

Loop II: (1) (4) (5) (6)

Loop I:

input: θ_2

output: r_3 θ_4

$(\theta_3 + \frac{\pi}{2} = \theta_4)$

Velocity

input: $\dot{\theta}_2$

output: $\dot{r}_3$ $\dot{\theta}_4$

$(\dot{\theta}_3 = \dot{\theta}_4)$

Loop II:

input: θ_4

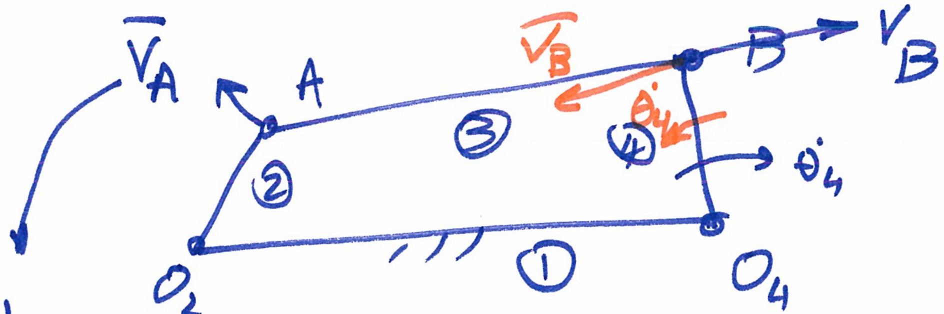
output: θ_5 r_6

HW 6.56

input: $\dot{\theta}_4$

output: $\dot{\theta}_5$ $\dot{r}_6$

Programming Velocity Analysis of Machines using Matlab



mag: $|r_2 \dot{\theta}_2|$

direction $\perp$ ②

$\boxed{HW}$ Write a code to analyze the velocity of Figure P4-3

Output: velocity of links ③ & ④

Create a report (PDF)