Chapter 4
Position (Displacement) Analysis

Why ??

- Understand the displacement of a machine at any time (extreme positions may not be sufficient to understand the motion)
- Understand the dynamics of a machine

$$F = m\,a$$

$$\downarrow$$

$$\frac{dv}{dt}$$

$$\downarrow$$

$$\frac{dx}{dt}$$

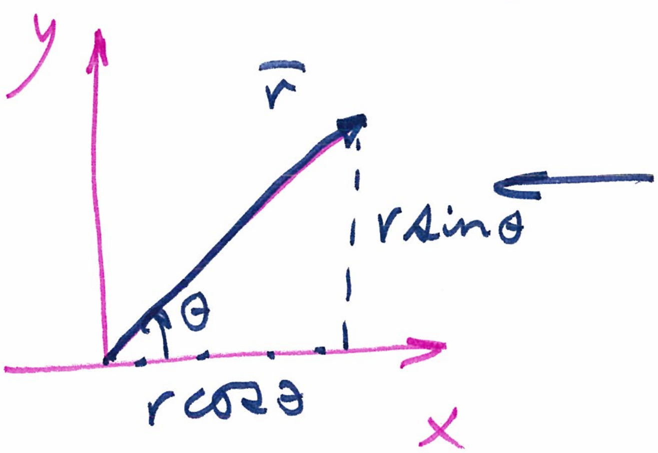
Background

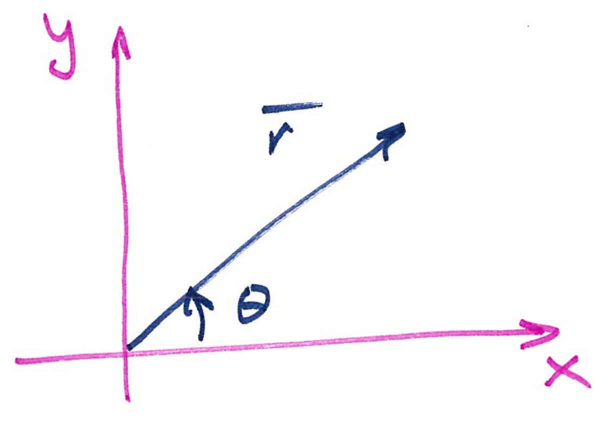
$j = \sqrt{-1}$

$e^{j\theta} = \cos\theta + j\sin\theta$ → DeMoivre Theorem

Alternative method for describing displacement (position) (2-D planar)

Let y-axis be associated with imaginary number j ($= \sqrt{-1}$)





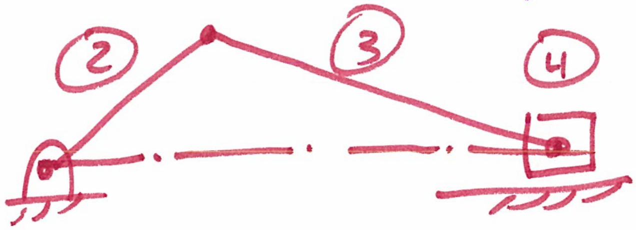
$\bar{r} = r(\cos\theta + j\sin\theta)$

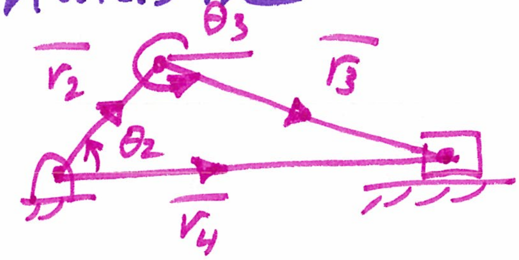
magnitude

$= r e^{j\theta}$ → Advantage: combine x & y components

complex phasor

Example: Use complex phasors to analyze position of crank-slider mechanism





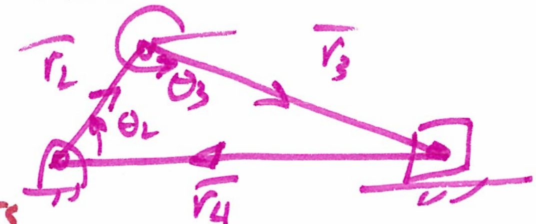
$\theta_4 = 0$

1. Associate all links with vectors

All angles should be measured CCW with respect to x-axis

2. Create a vector loop equation

$$\bar{r}_2 + \bar{r}_3 = \bar{r}_4 \longrightarrow \text{Alternative}$$



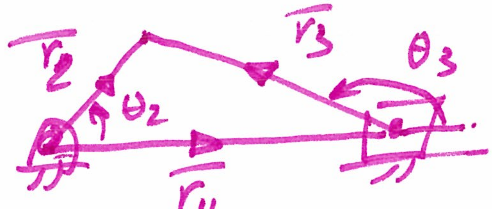
$\theta_4 = \pi$ $\quad \bar{r}_2 + \bar{r}_3 + \bar{r}_4 = 0$

$\boxed{\text{OR}}$

3. Write the vector loop equation in terms of complex phasors

$$r_2 e^{j\theta_2} + r_3 e^{j\theta_3} = r_4 e^{j\theta_4}$$

$$r_2 e^{j\theta_2} + r_3 e^{j\theta_3} = r_4$$



$\theta_4 = 0$

$\bar{r}_2 = \bar{r}_4 + \bar{r}_3$

4. Identify input & outputs

input: θ_2

output: θ_3 r_4

All angles should be measured at the start of the vector

(5) Replace $e^{j\theta}$ by $\cos\theta + j\sin\theta$

$$r_2(\cos\theta_2 + j\sin\theta_2) + r_3(\cos\theta_3 + j\sin\theta_3) = r_4$$

(6) Separate into real and imaginary equations

real $\rightarrow$ $r_2\cos\theta_2 + r_3\cos\theta_3 = r_4$ (1)

imag. $\rightarrow$ $r_2\sin\theta_2 + r_3\sin\theta_3 = 0$ (2)

$a + jb = 0$
$\downarrow$
$a = 0$, $b = 0$ } solution
coefficients

$(a+c) + j(b+d) = 0$
$a + c = 0$
$b + d = 0$

(7) Solve for the outputs in terms of the inputs & fixed terms

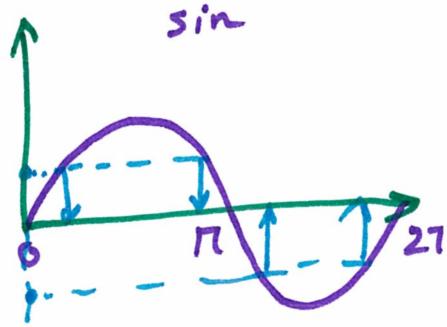
From (2)

$$r_3\sin\theta_3 = -r_2\sin\theta_2$$

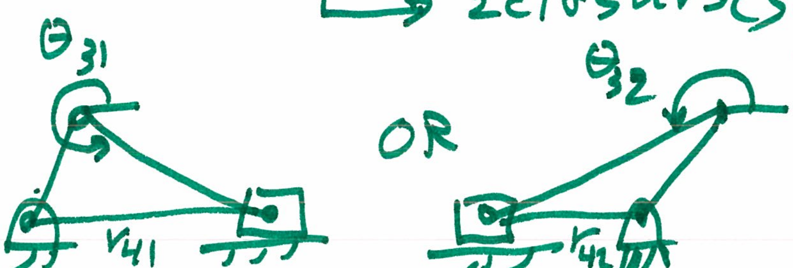
$$\sin\theta_3 = \frac{-r_2\sin\theta_2}{r_3}$$

$$\theta_3 = \sin^{-1}\left(-\frac{r_2\sin\theta_2}{r_3}\right)$$

$\rightarrow$ 2 values
$\rightarrow$ 2 closurses: θ_{31}, θ_{32}



$\sin^{-1} \rightarrow$ gives 2 answers

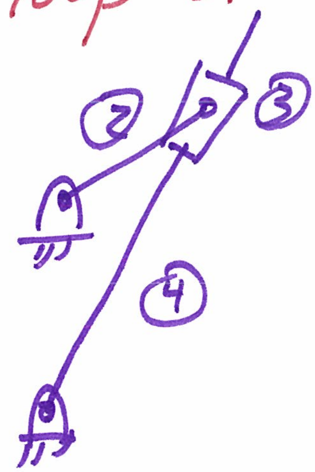


substitute in (1)

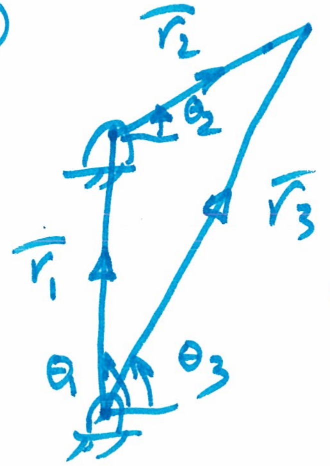
$$r_{41} = r_2\cos\theta_2 + r_3\cos\theta_{31}$$

$$r_{42} = r_2\cos\theta_2 + r_3\cos\theta_{32}$$

Example: Displacement Analysis of the first loop of the shaper QRM



(1)



$\theta_1 = \frac{\pi}{2}$

(2) $\overline{r_1} + \overline{r_2} = \overline{r_3}$

(3) $r_1 e^{j\theta_1} + r_2 e^{j\theta_2} = r_3 e^{j\theta_3}$

(4) input: θ_2

output: θ_3 r_3

(5) $r_1 (\cos\theta_1 + j\sin\theta_1) + r_2(\cos\theta_2 + j\sin\theta_2)$
$= r_3(\cos\theta_3 + j\sin\theta_3)$

$r_1(\cos\frac{\pi}{2} + j\sin\frac{\pi}{2}) + r_2(\cos\theta_2 + j\sin\theta_2)$
$= r_3(\cos\theta_3 + j\sin\theta_3)$

$$r_1(0+j) + r_2(\cos\theta_2 + j\sin\theta_2) = r_3(\cos\theta_3 + j\sin\theta_3)$$

6 real $\rightarrow$ $r_2\cos\theta_2 = r_3\cos\theta_3$ (1)

imag. $\rightarrow$ $r_1 + r_2\sin\theta_2 = r_3\sin\theta_3$ (2)

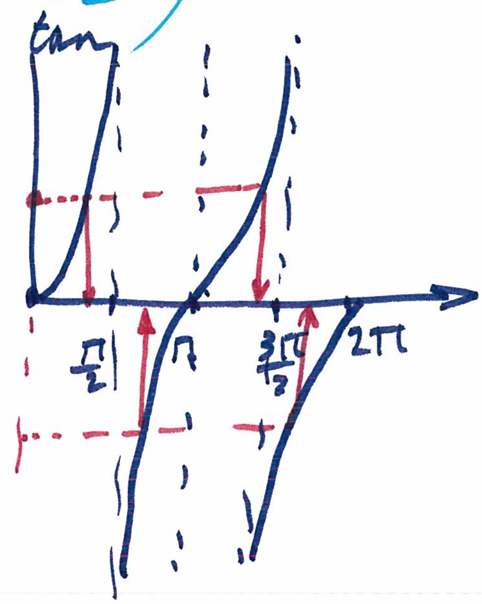
7 Divide (2) by (1) to eliminate r_3

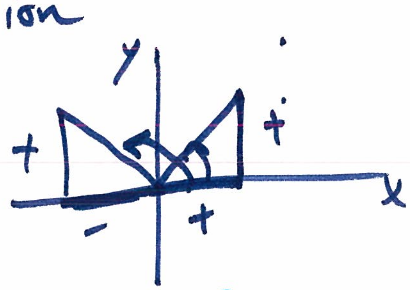
$$\frac{\cancel{r_3}\sin\theta_3}{\cancel{r_3}\cos\theta_3} = \frac{r_1 + r_2\sin\theta_2}{r_2\cos\theta_2}$$

$$\tan\theta_3 = \frac{r_1 + r_2\sin\theta_2}{r_2\cos\theta_2}$$

$$\theta_3 = \tan^{-1}\left(\frac{r_1 + r_2\sin\theta_2}{r_2\cos\theta_2}\right) \rightarrow \text{unique answer}$$

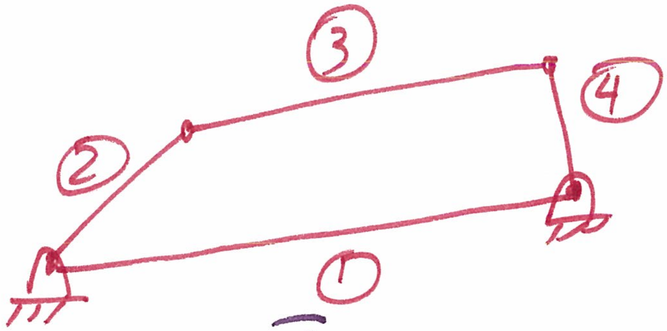
If both sides of the fraction are defined, $\tan^{-1}$ will yield a $\boxed{\text{unique}}$ solution

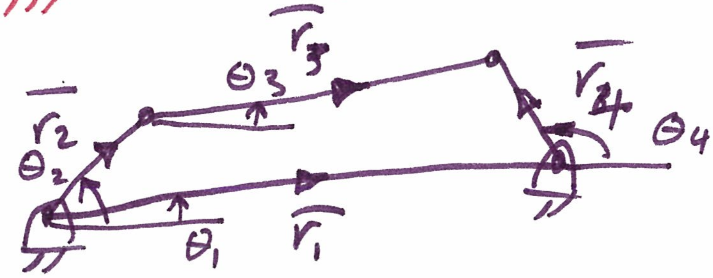




(1) $\rightarrow$ $r_3 = \dfrac{r_2\cos\theta_2}{\cos\theta_3}$

Displacement Analysis of a 4-Bar Mechanism



1 

2 $\overline{r_2} + \overline{r_3} = \overline{r_1} + \overline{r_4}$

3 $r_2 e^{j\theta_2} + r_3 e^{j\theta_3} = r_1 e^{j\theta_1} + r_4 e^{j\theta_4}$

4 input: θ_2
output: $\theta_3 \quad \theta_4$

5 $r_2(\cos\theta_2 + j\sin\theta_2) + r_3(\cos\theta_3 + j\sin\theta_3)$
$= r_1(\cos\theta_1 + j\sin\theta_1) + r_4(\cos\theta_4 + j\sin\theta_4)$

6 real $\rightarrow$ $r_2\cos\theta_2 + r_3\cos\theta_3 = r_1\cos\theta_1 + r_4\cos\theta_4$ (1)
imag. $\rightarrow$ $r_2\sin\theta_2 + r_3\sin\theta_3 = r_1\sin\theta_1 + r_4\sin\theta_4$ (2)

(7) Eliminate θ_3

$$\begin{cases} r_3\cos\theta_3 = r_1\cos\theta_1 + r_4\cos\theta_4 - r_2\cos\theta_2 \\ r_3\sin\theta_3 = r_1\sin\theta_1 + r_4\sin\theta_4 - r_2\sin\theta_2 \end{cases}$$

Squaring & adding

$$\boxed{\sin^2\theta + \cos^2\theta = 1}$$

$$r_3\cos\theta_3 = A + r_4\cos\theta_4$$

$$r_3\sin\theta_3 = B + r_4\sin\theta_4$$

where $A = r_1\cos\theta_1 - r_2\cos\theta_2$

$B = r_1\sin\theta_1 - r_2\sin\theta_2$

non θ_4 terms

$$r_3^2\cos^2\theta_3 + r_3^2\sin^2\theta_3 = (A + r_4\cos\theta_4)^2 + (B + r_4\sin\theta_4)^2$$

$$r_3^2 = A^2 + 2Ar_4\cos\theta_4 + \boxed{r_4^2\cos^2\theta_4} + B^2 + 2Br_4\sin\theta_4 + \boxed{r_4^2\sin^2\theta_4}$$

$$r_3^2 = A^2 + 2Ar_4\cos\theta_4 + B^2 + 2Br_4\sin\theta_4 + r_4^2$$

Rearranging

$$2Ar_4\cos\theta_4 + 2Br_4\sin\theta_4 = r_3^2 - A^2 - B^2 - r_4^2 \quad (3)$$

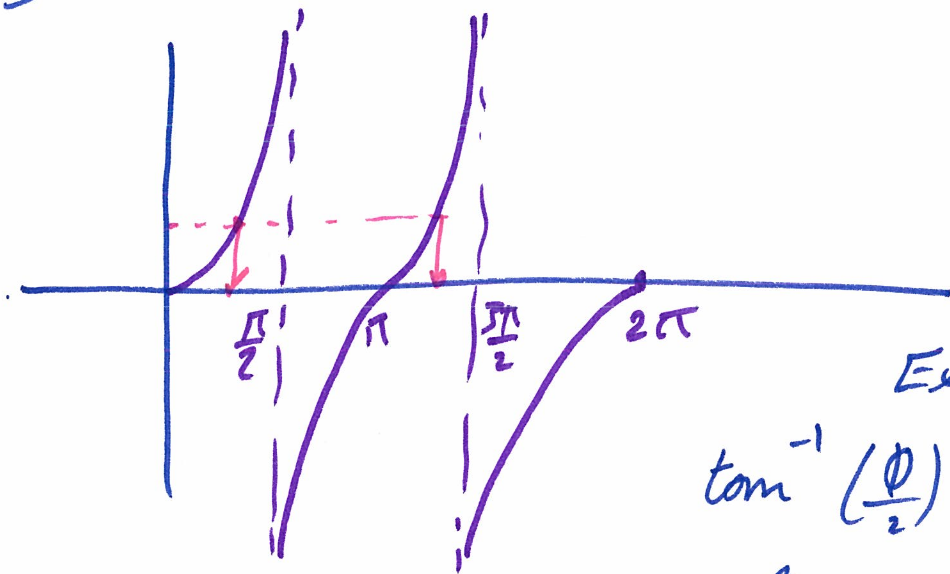
1 equation

1 unknown : θ_4

Half-Tangent Rule to solve equation in the form of:

$$a\cos\phi + b\sin\phi = c$$

Why do we use half tangent?



Example

$$\tan^{-1}\left(\frac{\phi}{2}\right) = 1$$

$$\frac{\phi}{2} = 45^\circ \text{ or } 225^\circ$$

$$\phi = 90^\circ \text{ or } 450^\circ$$

$$\downarrow$$

$$\underbrace{360 + 90^\circ}_{\text{unique solution}}$$

Because $\tan^{-1}\left(\frac{\phi}{2}\right)$ are 180° apart,

ϕ are 360° 〃

same angle

Solve ③ base on the Half-tangent rule

$\hookrightarrow$ obtain 2 values of θ_4:

θ_{4a} & $\theta_{4b} \longrightarrow$ 2 closures

Eliminate θ_4

$$r_4 \cos\theta_4 = r_2 \cos\theta_2 + r_3 \cos\theta_3 - r_1 \cos\theta_1$$
$$r_4 \sin\theta_4 = r_2 \sin\theta_2 + r_3 \sin\theta_3 - r_1 \sin\theta_1$$

$$r_4 \cos\theta_4 = C + r_3 \cos\theta_3$$
$$r_4 \sin\theta_4 = D + r_3 \sin\theta_3$$

where $C = r_2 \cos\theta_2 - r_1 \cos\theta_1$
$D = r_2 \sin\theta_2 - r_1 \sin\theta_1$

Squaring & adding

$$r_4^2 \cos^2\theta_4 = (C + r_3 \cos\theta_3)^2$$
$$r_4^2 \sin^2\theta_4 = (D + r_3 \sin\theta_3)^2$$

$$r_4^2 = C^2 + 2r_3 C \cos\theta_3 + r_3^2 \cos^2\theta_3 + D^2 + 2 r_3 D \sin\theta_3 + r_3^2 \sin^2\theta_3$$

Rearranging

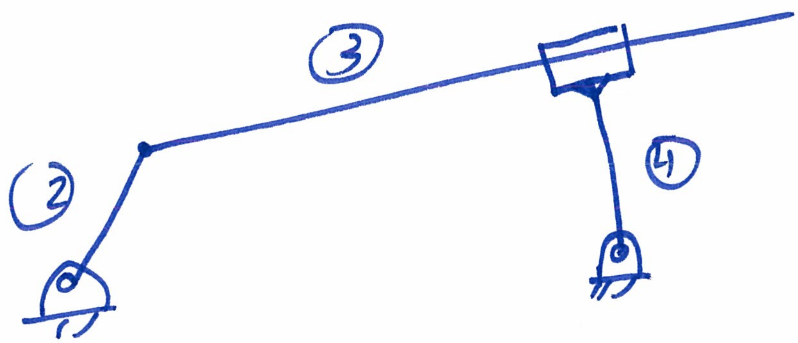
$$2 r_3 C \cos\theta_3 + 2 r_3 D \sin\theta_3 = r_4^2 - C^2 - D^2 - r_3^2$$

$\hookrightarrow$ solve using Half-tangent rule

θ_{3a} θ_{3b}

Example

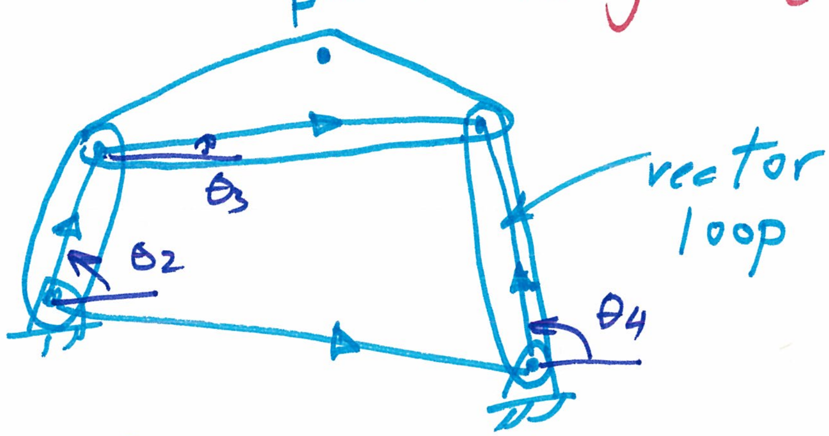
Perform steps ① through ⑥ of displacement analysis



$\boxed{H/W}$ 4.18 figure (f)

$\boxed{C}$

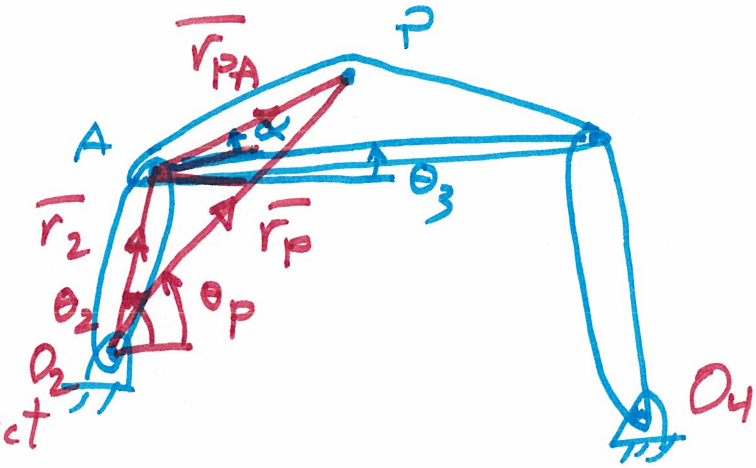
Position Analysis of a Point that Is not a Joint



* To analyze the position (displacement) of P, a secondary analysis is needed.

* The first step is to analyze position of the machine using vector loop & complex phasors approach → In this example, find θ_3 & θ_4 in terms of input (θ_2)

* Create a vector loop describing point P



* $\overline{r}_{PA}$ describes the relative position of P with respect to A

* α is the angle between $\overline{r}_3$ & $\overline{r}_{PA}$ (Fixed)

① $\overline{r}_2 + \overline{r}_{PA} = \overline{r}_P$

② $r_2 e^{j\theta_2} + r_{PA} e^{j(\theta_3+\alpha)} = r_P e^{j\theta_P}$

③ input: $\theta_2 \quad \theta_3$

output: $\theta_P \quad r_P$

④ $r_2(\cos\theta_2 + j\sin\theta_2) + r_{PA}(\cos(\theta_3+\alpha) + j\sin(\theta_3+\alpha$

$= r_P(\cos\theta_P + j\sin\theta_P)$

⑤ real $\rightarrow$ $r_2\cos\theta_2 + r_{PA}\cos(\theta_3+\alpha) = r_P\cos\theta_P$ ①

imag. $\rightarrow$ $r_2\sin\theta_2 + r_{PA}\sin(\theta_3+\alpha) = r_P\sin\theta_P$ ②

Squaring & adding

$$(r_2 \cos\theta_2 + r_{PA}\cos(\theta_3+\alpha))^2$$
$$+ (r_2 \sin\theta_2 + r_{PA}\sin(\theta_3+\alpha))^2$$
$$= r_p^2\cos^2\theta_p + r_p^2\sin^2\theta_p$$

exploiting $\sin^2 + \cos^2 = 1$ identity

$$r_2^2 + r_{PA}^2 + 2 r_2 r_{PA}\cos\theta_2\cos(\theta_3+\alpha)$$
$$+ 2 r_2 r_{PA}\sin\theta_2\sin(\theta_3+\alpha) = r_p^2$$

identity $\cos a \cos b + \sin a \sin b = \cos(a-b)$

$$r_2^2 + r_{PA}^2 + 2 r_2 r_{PA}\cos(\theta_3+\alpha-\theta_2) = r_p^2$$

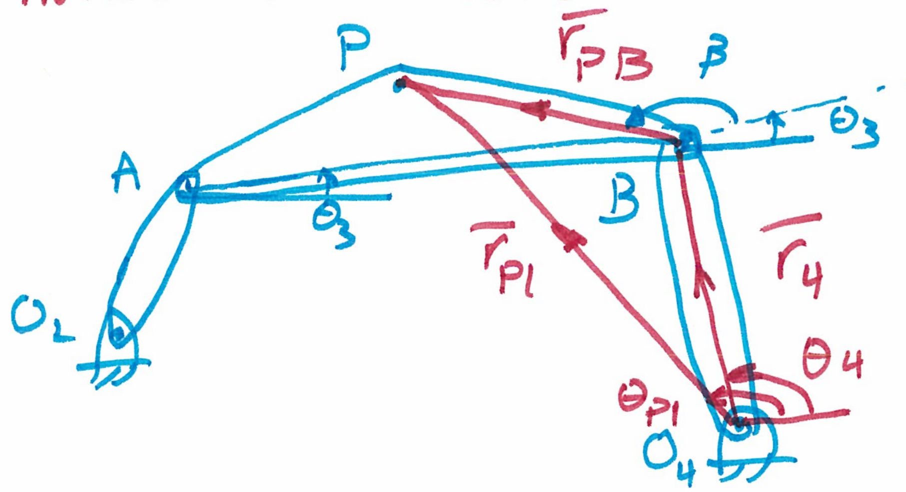
$$r_p = \left[r_2^2 + r_{PA}^2 + 2 r_2 r_{PA}\cos(\theta_3+\alpha-\theta_2)\right]^{1/2}$$

Dividing ② by ①

$$\tan\theta_p = \frac{(r_2\sin\theta_2 + r_{PA}\sin(\theta_3+\alpha))}{(r_2\cos\theta_2 + r_{PA}\cos(\theta_3+\alpha))}$$

use the signs of the two sides of the fraction to obtain a <u>unique</u> solution

Alternative Formulation



$$\bar{r}_{\cancel{4}} + \bar{r}_{PB} = \bar{r}_{P1}$$

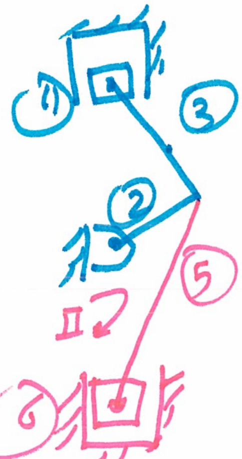
$$r_4 e^{j\theta_4} + r_{PB} e^{j(\theta_3 + \beta)} = r_{P1} e^{j\theta_{P1}}$$

$\vdots$

Position Analysis of Machines with Multiple Loops

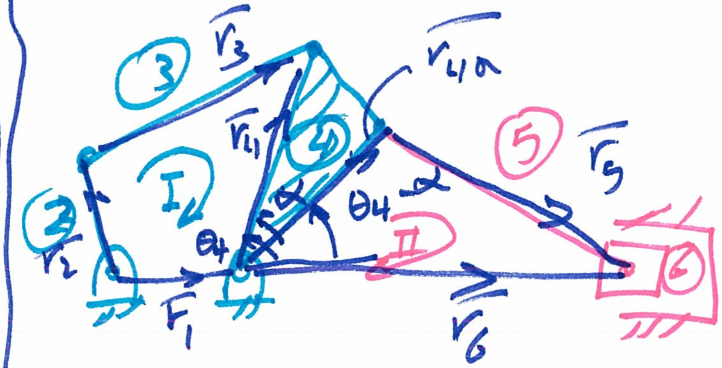
Loops

Parallel



* Loops can be solved separately

Series



* Loops have to be solved in the same sequence motion is transmitted

* In the example above

Loop I
input : θ_2
output : θ_3 θ_4

Loop II : $\cancel{\theta_4}$
input : θ_4
output : r_6 θ_5

HW 4.19 4.22 Due Tuesday

Computer Simulation