Chapter 11
Dynamic Force Analysis

Newton's 2nd Law

$$\Sigma F = m a$$

or

$$\Sigma M = I \alpha$$

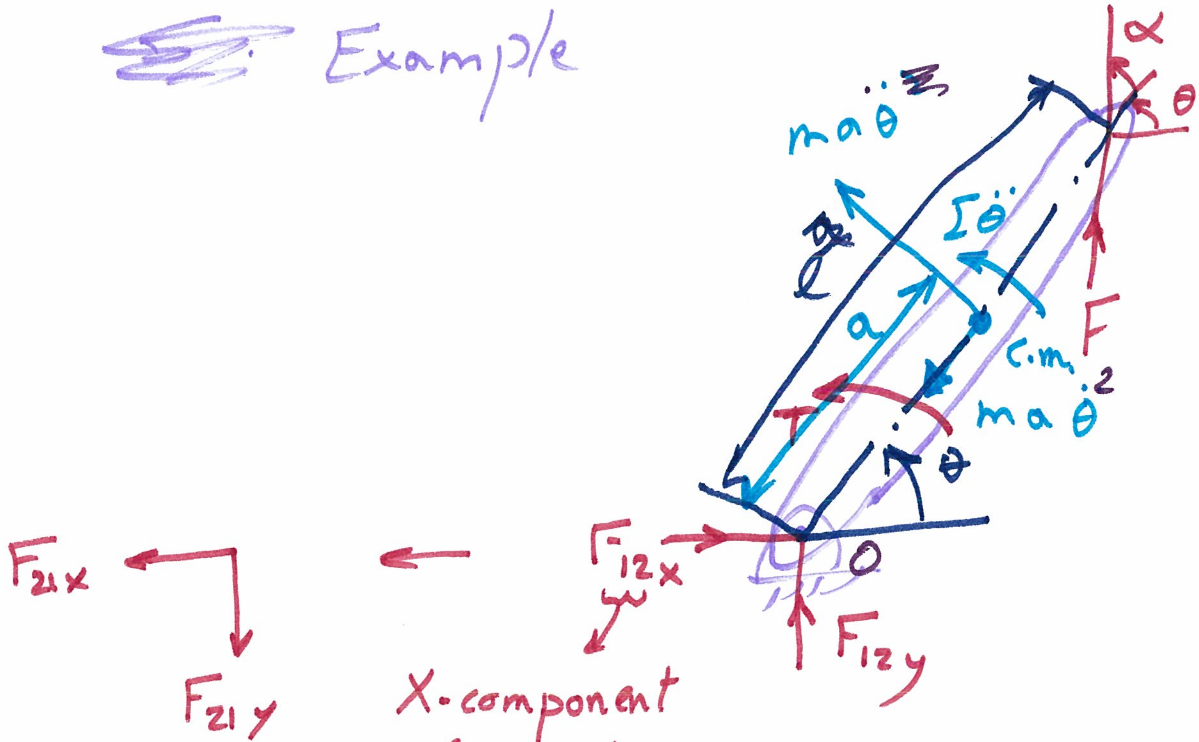
In plane,

$$\Sigma F_x = m a_x$$

$$\Sigma F_y = m a_y$$

$$\Sigma M_z = I \alpha$$

Example



X-component of reaction force between 1 & 2

Summary:

External loads: $T \quad F$

Reaction forces: $F_{12x} \quad F_{12y}$

Dynamic forces: $ma\ddot{\theta} \quad ma\dot{\theta}^2 \quad I\ddot{\theta}$

$\Sigma M_0 \longrightarrow$ (along the z-axis)

normal to the plane where the link moves

moment of external ~~loads~~ forces

$$\underbrace{T\bar{K} + \bar{\ell}\times\bar{F}}_{\text{moment of external forces}} = \underbrace{I\ddot{\theta}\bar{K} + (ma\ddot{\theta})a\bar{K}}_{\text{moment of dynamic forces}}$$

$$T\bar{K} + \begin{vmatrix} \bar{i} & \bar{j} & \bar{K} \\ \ell\cos\theta & \ell\sin\theta & 0 \\ F\cos(\theta+\alpha) & F\sin(\theta+\alpha) & 0 \end{vmatrix}$$

$$\boxed{\begin{vmatrix} \bar{i} & \bar{j} & \bar{K} \\ a & b & c \\ d & e & f \end{vmatrix}}$$

determinant

$bf - ec$

$$= I\ddot{\theta}\bar{K} + ma^2\ddot{\theta}\bar{K}$$

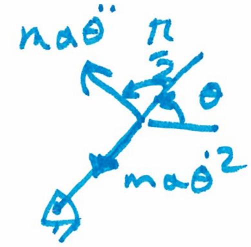
$$T\bar{K} + \left((\ell\cos\theta)(F\sin(\theta+\alpha)) - (\ell\sin\theta)(F\cos(\theta+\alpha))\right)\bar{K}$$
$$= I\ddot{\theta}\bar{K} + ma^2\ddot{\theta}\bar{K}$$

Canceling $\bar{K}$

$$T + F\ell(\cos\theta\sin(\theta+\alpha) - \sin\theta\cos(\theta+\alpha)) = I\ddot{\theta} + ma^2\ddot{\theta}$$

$$T + F\ell\sin((\theta+\alpha)-\theta) = (I + ma^2)\ddot{\theta}$$

$$T + F\ell\sin\alpha = \underbrace{(I + ma^2)}_{\text{parallel axis theorem}}\ddot{\theta}$$



ΣF_x

$$F_{12x} + F\cos(\theta+\alpha) = ma\ddot{\theta}\underbrace{\cos\left(\theta+\frac{\pi}{2}\right)}_{-\sin\theta} - ma\dot{\theta}^2\cos\theta$$

ΣF_y

$$F_{12y} + F\sin(\theta+\alpha) = ma\ddot{\theta}\overbrace{\sin\left(\theta+\frac{\pi}{2}\right)}^{\cos\theta} - ma\dot{\theta}^2\sin\theta$$

Assemble the 3 equations using matrix form (assume that F is known and we are interested in understanding the resulting motion)

$$\begin{array}{r} \Sigma M_0 \rightarrow \\ \Sigma F_x \rightarrow \\ \Sigma F_y \rightarrow \end{array} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{coefficent matrix}} \underbrace{\begin{Bmatrix} T \\ F_{12x} \\ F_{12y} \end{Bmatrix}}_{\text{external forces}} + \underbrace{\begin{Bmatrix} F\ell\sin\alpha \\ F\cos(\theta+\alpha) \\ F\sin(\theta+\alpha) \end{Bmatrix}}_{F \text{ terms}}$$

$$= \underbrace{\begin{Bmatrix} I + ma^2 \\ ma\cos\left(\theta+\frac{\pi}{2}\right) \\ ma\sin\left(\theta+\frac{\pi}{2}\right) \end{Bmatrix}}_{\ddot{\theta} \text{ coeff.}} \ddot{\theta} + \underbrace{\begin{Bmatrix} 0 \\ -ma\cos\theta \\ -ma\sin\theta \end{Bmatrix}}_{\dot{\theta}^2 \text{ coeff.}} \dot{\theta}^2$$

Interpretations

Given: ① dimensions and mass of the system

② External force (F, α), and location of F

We have 2 possible objectives

① If $\theta\ \dot{\theta}\ \ddot{\theta}$ is given (motion-time history), T can be calculated to create this motion.

F_{12x} F_{12y} are also calculated

$\boxed{\text{Inverse dynamic problem}}$

Rephrase the system of equations

& $$[A]\{F\} + \{F_e\} = \{B\}\ddot{\theta} + \{C\}\dot{\theta}^2$$

$$[A]\{F\} = \{B\}\ddot{\theta} + \{C\}\dot{\theta}^2 - \{F_e\}$$

$$\{F\} = [A]^{-1}\left(\{B\}\ddot{\theta} + \{C\}\dot{\theta}^2 - \{F_e\}\right)$$

→ algebraic equations

② If T is given (external torque), calculate the resulting motion $\ddot{\theta}$ $\dot{\theta}$ θ time histrory

$\boxed{\text{Direct Dynamic Problem}}$

Go to the first equation (no reaction forces)

$$T + F l \sin\alpha = (I + m a^2)\ddot{\theta} + 0$$

→ differential equation

↓

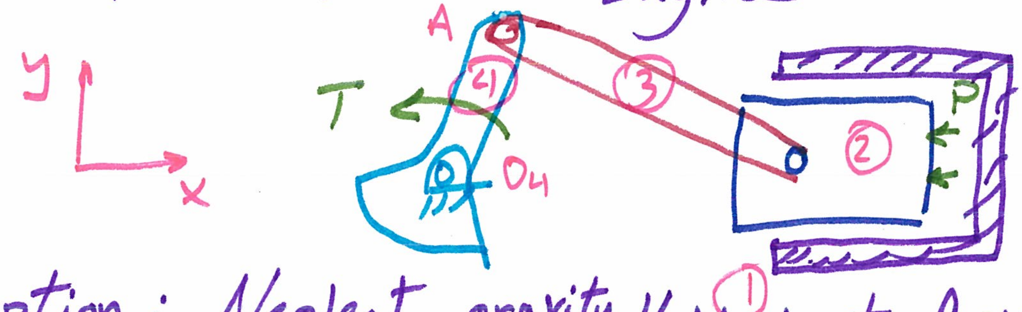
$\dot{\theta}$ & θ can be obtained by integrating the equation & applying initial conditions

Go to equations ② & ③ (ΣF_x & ΣF_y) to obtain F_{12x} F_{12y} time histories

→ In general, the system is in the form of nonlinear d.e. because of the $\dot{\theta}^2$

Example:

Internal Combustion Engine

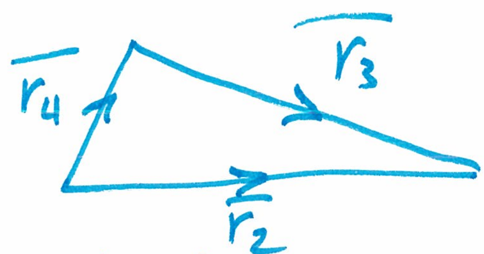


Assumption: Neglect gravity effects // Neglect friction forces

T: External torque (load)

P: Force due to combustion

Vector Loop



$\overline{r_2} = \overline{r_3} + \overline{r_4}$

↓

Displacement, velocity, acceleration analysis $\longrightarrow$ $\left.\begin{matrix} \theta_3 & \theta_4 \\ \dot{\theta}_3 & \dot{\theta}_4 \\ \ddot{\theta}_3 & \ddot{\theta}_4 \end{matrix}\right\}$ in terms of $r_2 \;\; \dot{r}_2 \;\; \ddot{r}_2$

Approach:

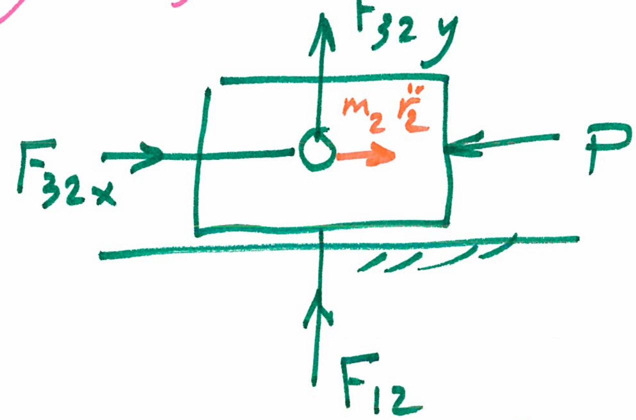
- Free body diagram for each link
- Develop dynamic balance equations " " "
- Assemble the equations in a matrix form

Free-Body Diagrams

2

2 equations

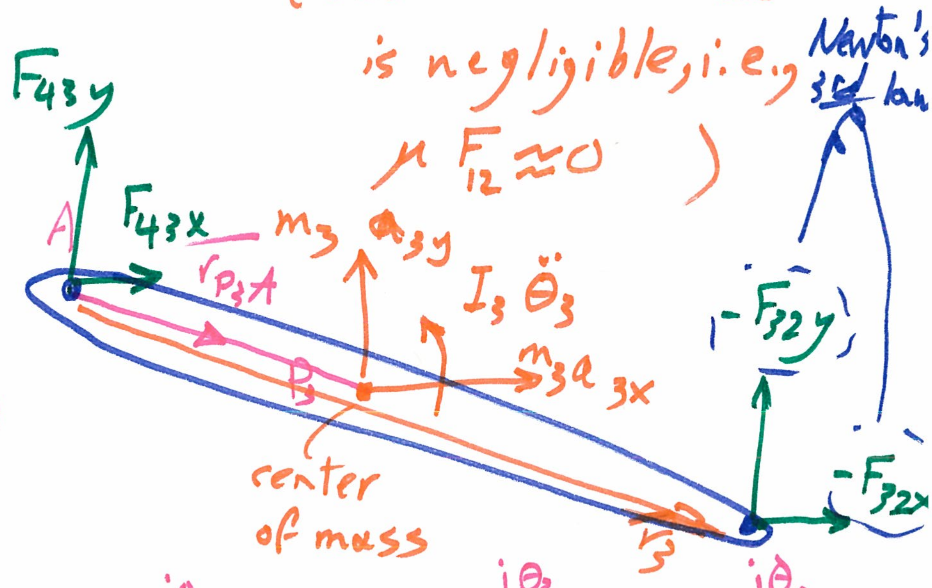
3 unknowns



(assume friction is negligible, i.e., $\mu F_{12} \approx 0$)

3

a_3: acceleration of the center of mass of link 3



$$\bar{a}_3 = \bar{a}_A + \bar{a}_{P_3A}$$

$$= r_4 \ddot{\theta}_4 j e^{j\theta_4} - r_4 \dot{\theta}_4^2 e^{j\theta_4} + r_{P_3A} \ddot{\theta}_3 j e^{j\theta_3} - r_{P_3A} \dot{\theta}_3^2 e^{j\theta_3}$$

3 equations

2 new unknowns

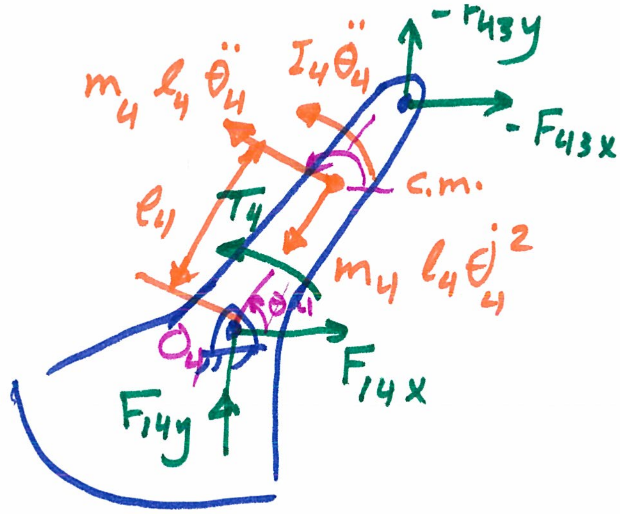
④

3 equations

3 new unknowns

$\downarrow$

T_4 F_{14y} F_{14x}



ΣM_{O_4}

$$T_4 \bar{k} + \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ r_4 \cos\theta_4 & r_4 \sin\theta_4 & 0 \\ -F_{43x} & -F_{43y} & 0 \end{vmatrix}$$

$$= I_4 \ddot{\theta}_4 \bar{k} + \underbrace{m_4 l_4 \ddot{\theta}_4}_{\text{normal to the link}} (l_4) \bar{k}$$

$$T_4 \bar{k} + (-F_{43y} r_4 \cos\theta_4 + F_{43x} r_4 \sin\theta_4) \bar{k} = I_4 \ddot{\theta}_4 \bar{k} + m_4 l_4^2 \ddot{\theta}_4 \bar{k}$$

Solve the system for inverse dynamic problem (motion is known)

$$\underbrace{[A]}_{\text{coefficients}} \underbrace{\{q\}}_{\text{force unknowns}} = \underbrace{\{B\}}_{\text{dynamic forces + input forces}}$$

$$\{q\} = [A]^{-1} \{B\}$$

Dynamic Balance Equations
(Newton's 2nd Law)

② ΣF_x $\quad F_{32x} - P = m_2 \ddot{r}_2$

ΣF_y $\quad F_{32y} + F_{12} = 0$

③ ΣF_x $\quad -F_{32x} + F_{43x} = m_3 a_{3x}$

ΣF_y $\quad -F_{32y} + F_{43y} = m_3 a_{3y}$

ΣM_A

[origin of the vector $\bar{r}_3$.]

$$\begin{vmatrix} \bar{i} & \bar{j} & \bar{K} \\ r_3 \cos\theta_3 & r_3 \sin\theta_3 & 0 \\ -F_{32x} & -F_{32y} & 0 \end{vmatrix} = I_3 \ddot{\theta}_3 \bar{K} + \begin{vmatrix} \bar{i} & \bar{j} & \bar{K} \\ r_{P_3A} \cos\theta_3 & r_{P_3A} \sin\theta_3 & 0 \\ m_3 a_{3x} & m_3 a_{3y} & 0 \end{vmatrix}$$

$$(-r_3 \cos\theta_3 F_{32y} + r_3 \sin\theta_3 F_{32x})\bar{K}$$
$$= I_3 \ddot{\theta}_3 \bar{K} + (m_3 a_{3y} r_{P_3A} \cos\theta_3 - m_3 a_{3x} r_{P_3A} \sin\theta_3)\bar{K}$$

④ ΣF_x $\quad -F_{43x} + F_{14x} = m_4 l_4 \ddot{\theta}_4 \cos(\theta_4 + \frac{\pi}{2}) - m_4 l_4 \dot{\theta}_4^2 \cos\theta_4$

ΣF_y $\quad -F_{43y} + F_{14y} = m_4 l_4 \ddot{\theta}_4 \sin(\theta_4 + \frac{\pi}{2}) - m_4 l_4 \dot{\theta}_4^2 \sin\theta_4$

ΣM_{O_4}

Apply to the example

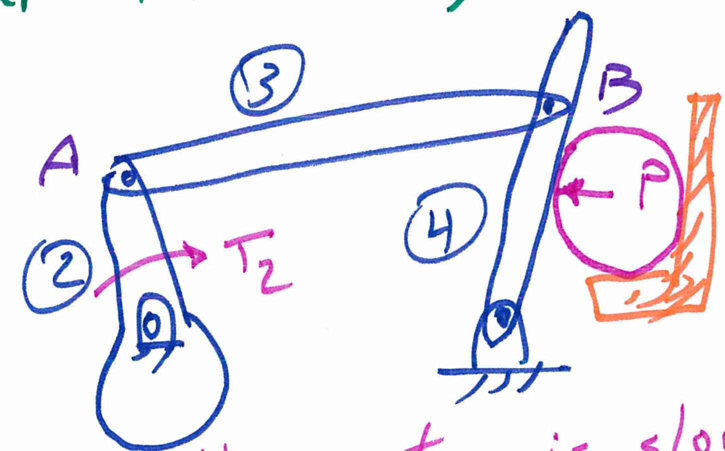
$$
\begin{array}{c}
2 \\ \\ \\ 3 \\ \\ \\ 4 \\ \\
\end{array}
\begin{bmatrix}
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\
0 & r_3 \sin\theta_3 & -r_3 \cos\theta_3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\
0 & 0 & 0 & 0 & r_4 \sin\theta_4 & -r_4 \cos\theta_4 & 0 & 0
\end{bmatrix}
\begin{Bmatrix}
P \\ F_{32x} \\ F_{32y} \\ F_{12} \\ F_{43x} \\ F_{43y} \\ F_{14x} \\ F_{14y} \\ \cancel{T_4}
\end{Bmatrix}
$$

$$
= \begin{Bmatrix}
m_2 \ddot{r}_2 \\
0 \\
m_3 a_{3x} \\
m_3 a_{3y} \\
I_3 \ddot{\theta}_3 + m_3 a_{3y} r_{P_3A} \cos\theta_3 - m_3 a_{3x} r_{P_3A} \sin\theta_3 \\
m_4 l_4 \ddot{\theta}_4 \cos(\theta_4 + \frac{\pi}{2}) - m_4 l_4 \dot{\theta}_4^2 \cos\theta_4 \\
m_4 l_4 \ddot{\theta}_4 \sin(\theta_4 + \frac{\pi}{2}) - m_4 l_4 \dot{\theta}_4^2 \sin\theta_4 \\
I_4 \ddot{\theta}_4 + m_4 l_4^2 \ddot{\theta}_4 - T_4
\end{Bmatrix}
$$

(Assuming that T_4 and motion are known. The objective is to calculate P and reaction forces)

H.W.

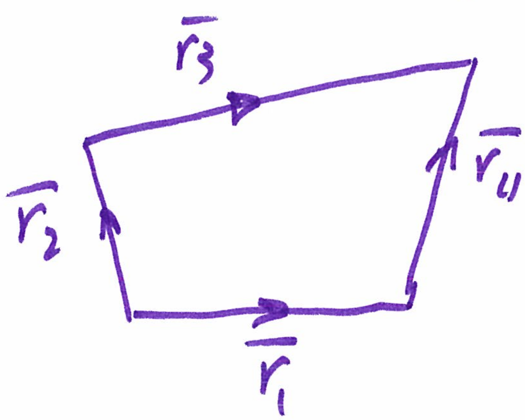
Stone Crusher Force Analysis



* Assume that the motion is slow (dynamic forces are negligible)
* Gravity load exists

Find relation between P and T_2

given

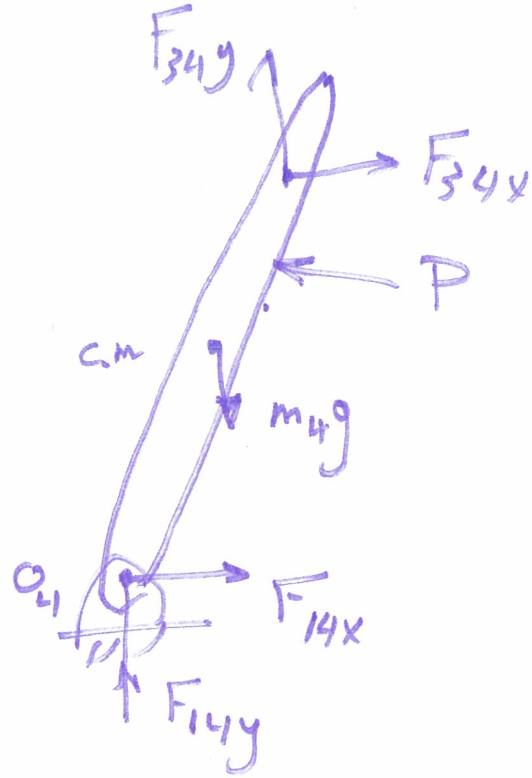


4

ΣF_x
ΣF_y
ΣM_{O_4}

$\left.\begin{array}{l}\Sigma F_x \\ \Sigma F_y \\ \Sigma M_{O_4}\end{array}\right\}$ 3 equations

4 unknowns



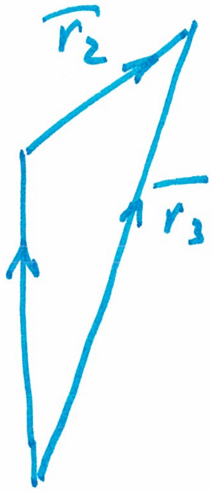
3

2

Assemble the equations

9 equation

Vector Loop

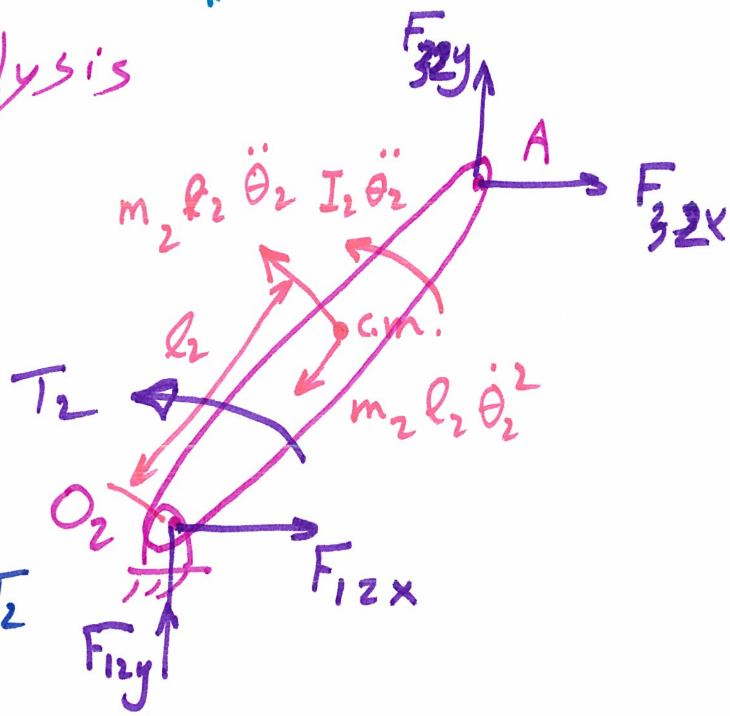


solve for r_3 θ_3 in terms of θ_2
" " $\dot{r}_3$ $\dot{\theta}_3$ " " $\dot{\theta}_2$
" " $\ddot{r}_3$ $\ddot{\theta}_3$ " " $\ddot{\theta}_2$

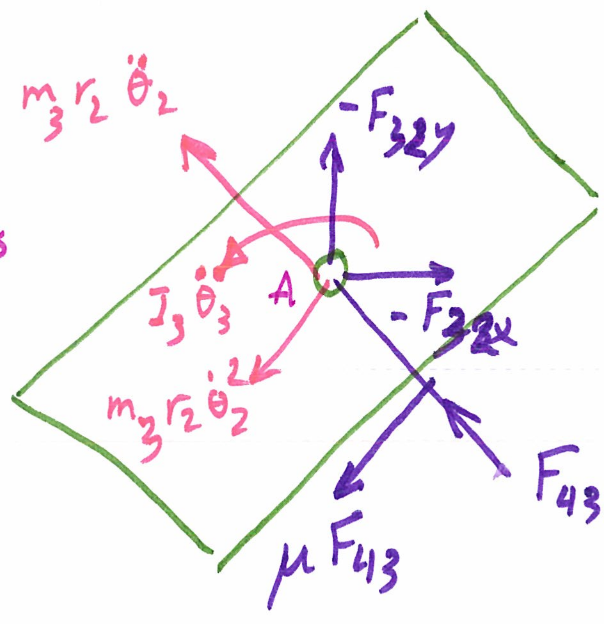
→ Kinematic analysis

② ΣF_x
ΣF_y
ΣM_{O_2}

5 unknowns:
F_{12x} F_{12y} F_{23x} F_{23y} T_2



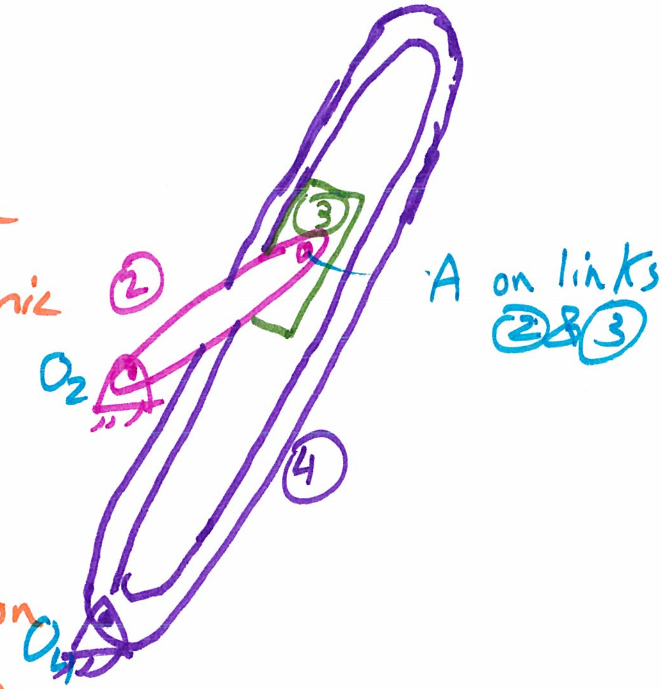
③ Assume that the hinge coincides with the center of mass



ΣF_x
ΣF_y
ΣM_A

1 new unknowns:

Dynamic Analysis of a Quick-Return Machine



* Solve the problem as an inverse dynamic problem: Given the motion find T_2 that achieves this motion and reaction forces

Assumptions:

* Motion is fast → dynamic forces cannot be neglected
* Gravity force are negligible
* There is friction between links 3 & 4

4

3 equations

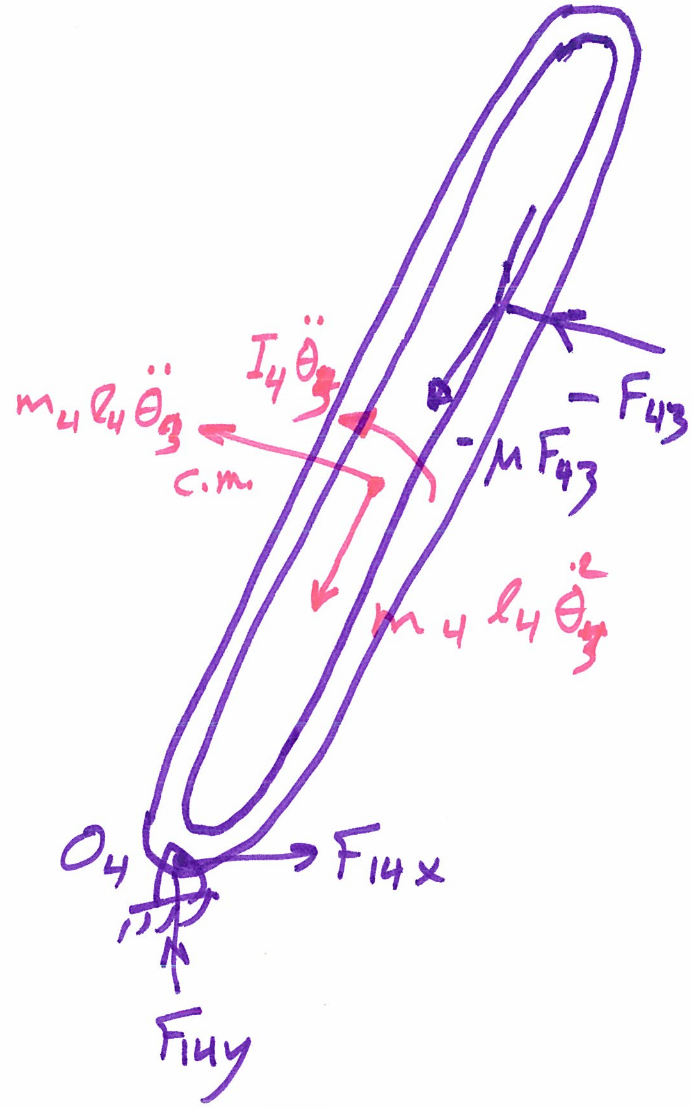
ΣF_x

ΣF_y

ΣM_{O_4}

2 new unknowns

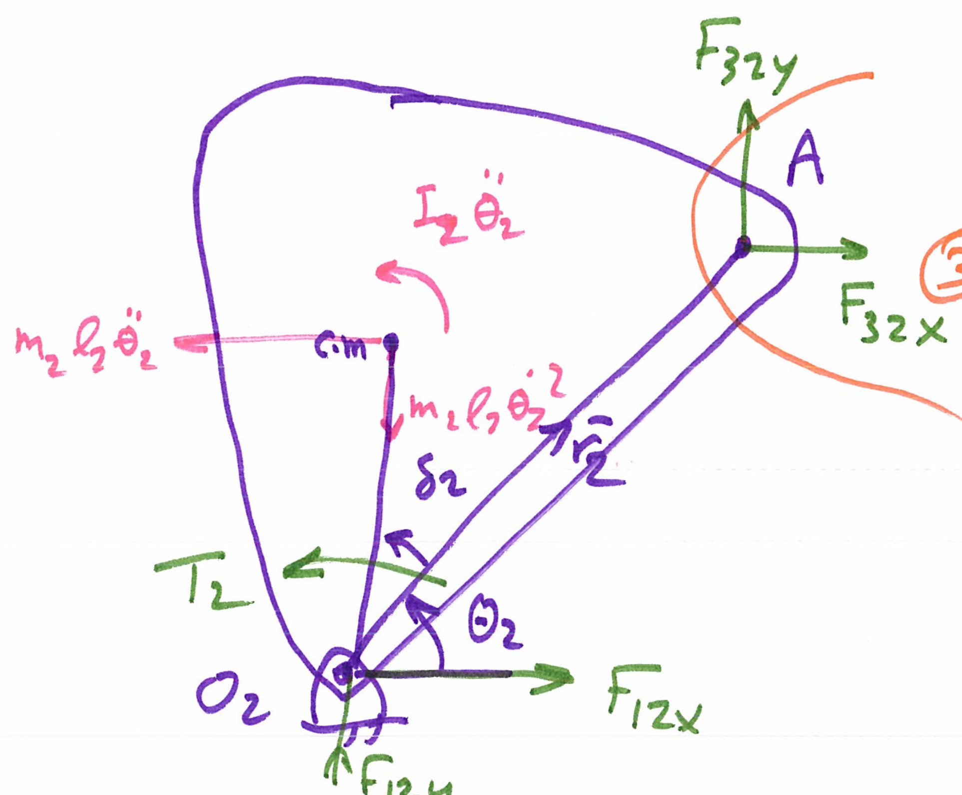
F_{14x} F_{14y}



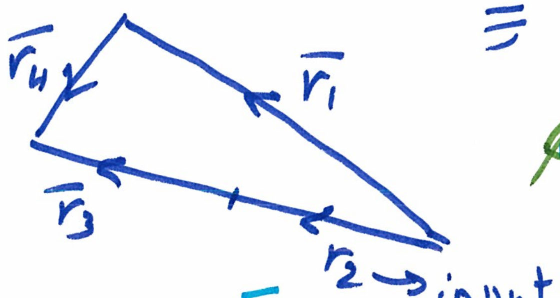
HW.

11.3 (a)

link 2



H.W
11.14
11.18



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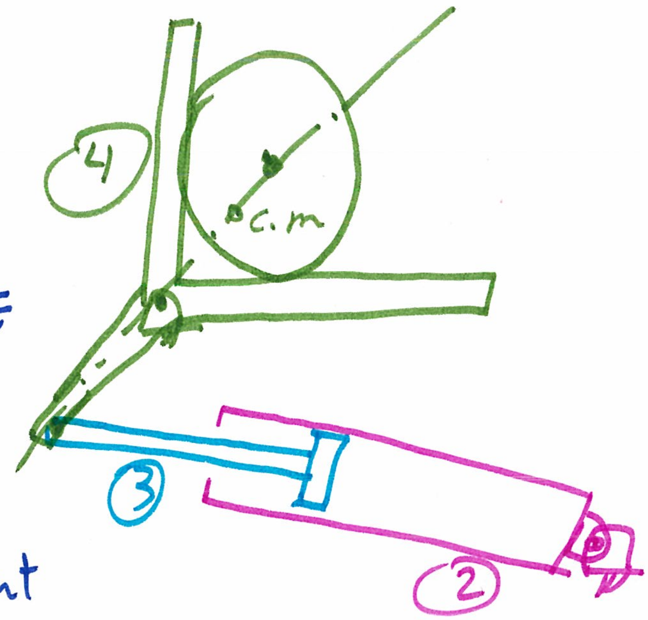

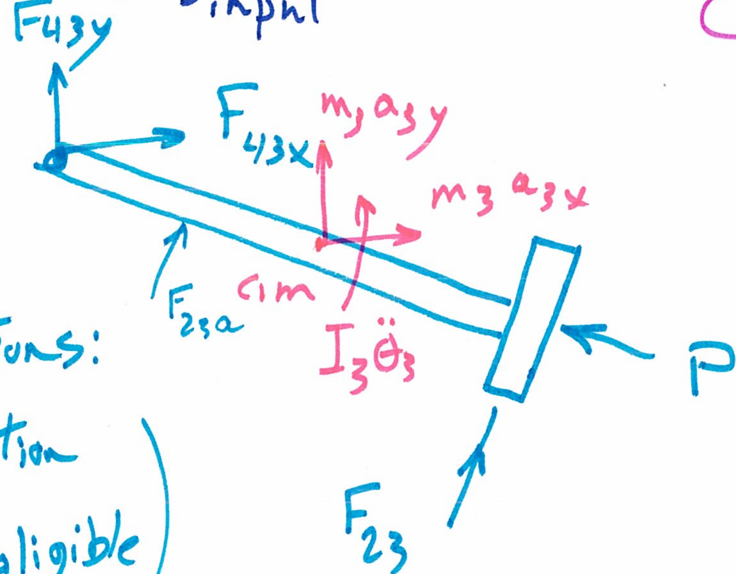

Assmptions:

(friction is negligible)

(weight of the piston/cylinder is small compared to the weight of link 4)