Chapter 2 Review

Note Title 2/7/2008

Laplace

$$\text{outp} - \ddot{y} + a\dot{y} + b\,y(t) = \underset{\substack{\uparrow\\ \text{in}}}{r(t)}$$

$\downarrow \quad s = \frac{d}{dt}$

$$T(s) = \frac{out(s)}{inp.(s)} = \frac{Y(s)}{R(s)} = \frac{1}{s^2 + as + b}$$

no initial values

$T(s) \Rightarrow$ partial fractions

Sol.: $y(t)$ from Table lookup

$$X(s) = \frac{(s+5)(s+3)}{s(s+2)(s+6)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+6}$$

A, B, C = "Residues"

$$X(s) = \frac{C_1}{s} + \frac{C_2}{s+2} + \frac{C_3}{s+6}.$$

$$C_1 = \left.\frac{(s+5)(s+3)}{(s+2)(s+6)}\right|_{s=0} = \frac{(5)(3)}{(2)(6)} = \frac{5}{4}$$

$$C_2 = \left.\frac{(s+5)(s+3)}{s(s+6)}\right|_{s=-2} = \frac{(3)(1)}{(-2)(4)} = -\frac{3}{8}$$

$$C_3 = \left.\frac{(s+5)(s+3)}{s(s+2)}\right|_{s=-6} = \frac{(-1)(-3)}{(-6)(-4)} = \frac{1}{8}$$

time domain solution

$$X(s) = \frac{\left(\frac{5}{4}\right)}{s} + \frac{\left(-\frac{3}{8}\right)}{s+2} + \frac{\left(\frac{1}{8}\right)}{s+6}$$

Final value $\lim_{t \to \infty} y(t) = \lim_{s \to 0} s \cdot F(s)$

use only for stable F(s)

Second Order Underdamped Systems

$$F(s) = \frac{Y(s)}{U(s)} = \frac{\omega^2}{s^2 + 2\cdot\zeta\cdot\omega\cdot s + \omega^2}$$

$0 < \underbrace{\zeta}_{= \text{Damping coeff.}} < 1$

ω = "Undamped nat. frequency

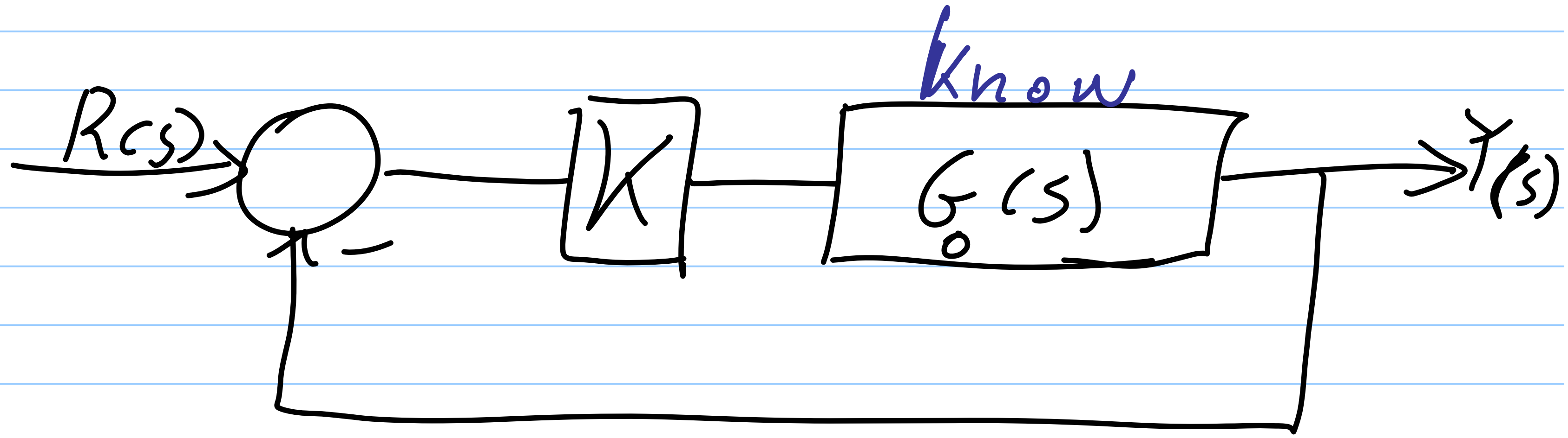
Characteristic Equation: DEN(s) = 0

Thus $s^2 + 2\cdot\zeta\cdot\omega\cdot s + \omega^2 = 0$

The solution vector is:

$$\begin{bmatrix} \left[-\zeta + \left(\zeta^2 - 1\right)^{\frac{1}{2}}\right]\cdot\omega \\ \left[-\zeta - \left(\zeta^2 - 1\right)^{\frac{1}{2}}\right]\cdot\omega \end{bmatrix}$$

Root locus method



$$\frac{Y}{R} = \frac{K\,G_0(s)}{1 + K \cdot G_0(s)}$$

Stability is solely determined by char. equation:

$$1 + K\,G_0(s) = 0$$

R locus: graphical solution of char. equ.

$$0 < K < \infty$$

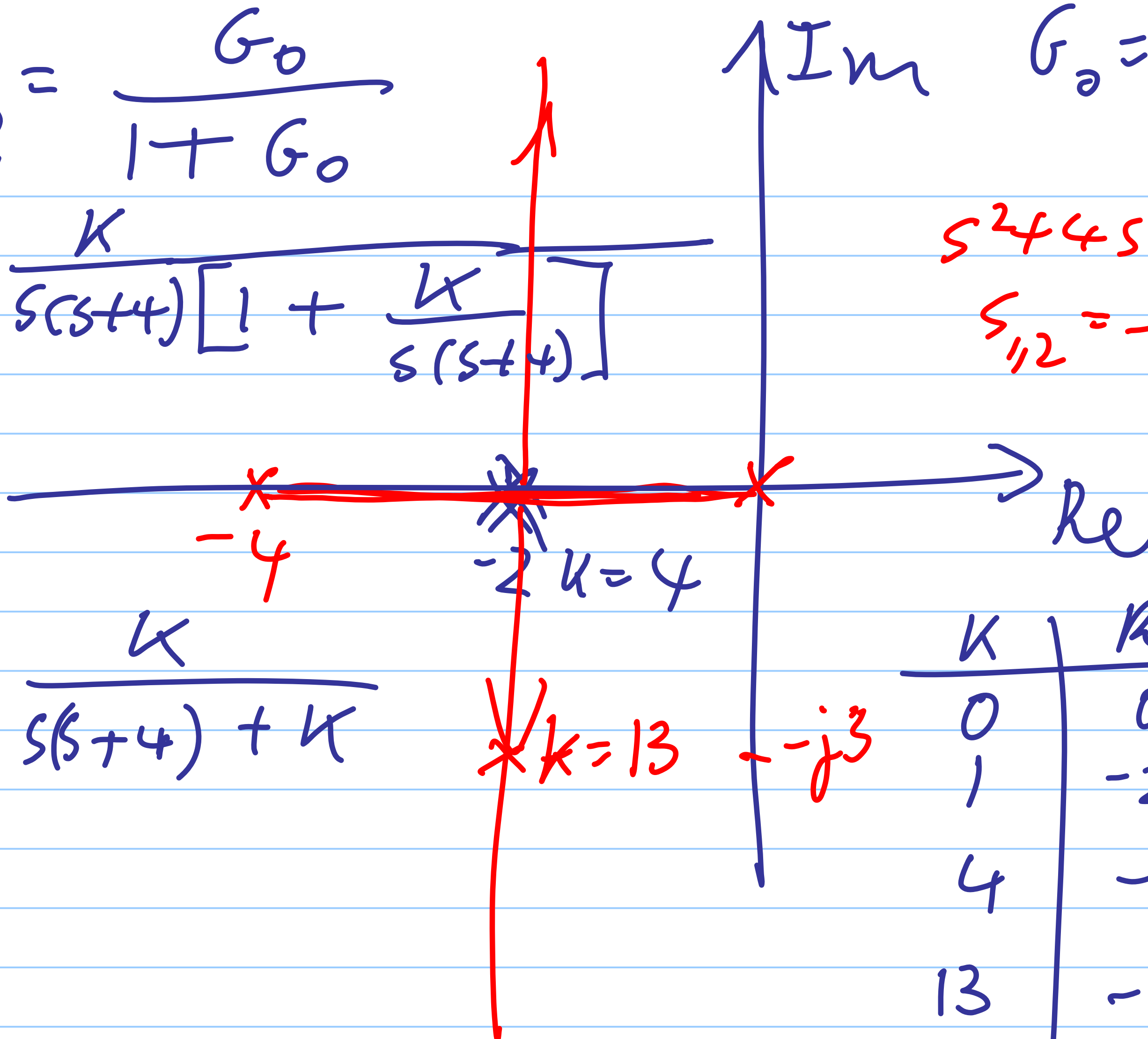
$$G_{closed} = \frac{G_o}{1 + G_o}$$

$$G_o = \frac{K}{s(s+4)}$$

$$= \frac{K}{s(s+4)\left[1 + \frac{K}{s(s+4)}\right]}$$

$$s^2 + 4s + K = 0$$

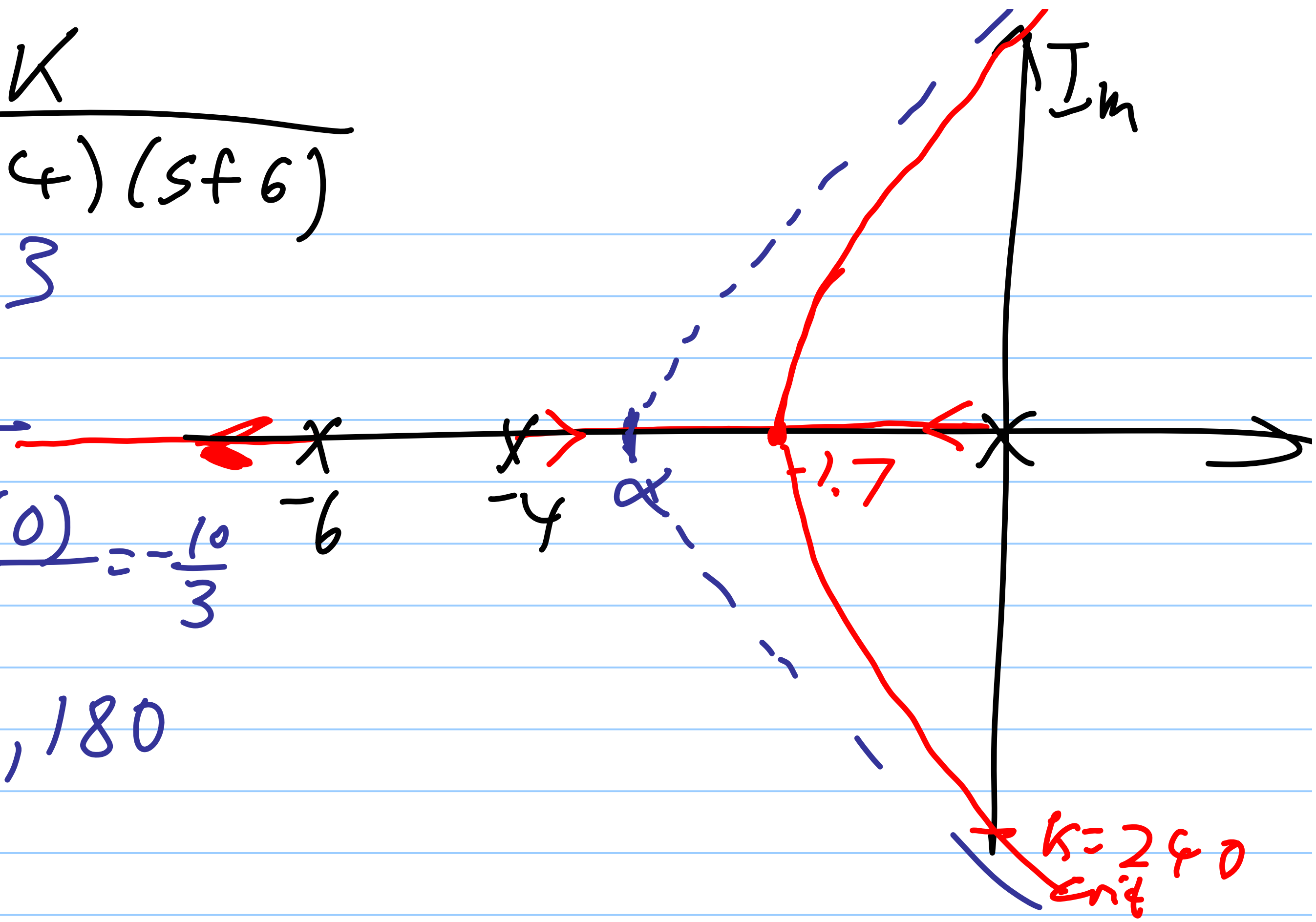
$$s_{1,2} = -2 \pm \sqrt{4 - K}$$



$$G_{closed} = \frac{K}{s(s+4) + K}$$

K	Roots
0	0, −4
1	$-2 \pm \sqrt{3}$
4	−2 double
13	$-2 \pm j3$

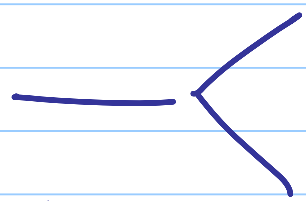
$$G_o(s) = \frac{K}{s(s+4)(s+6)}$$



$$p - z = 3 - 0 = 3$$

$$\alpha = \frac{\sum p_i - \sum z_i}{p - z} = \frac{0 - 4 - 6 - (0)}{3} = -\frac{10}{3}$$

$$\angle As = \pm 60, 180$$



X 4As

Im

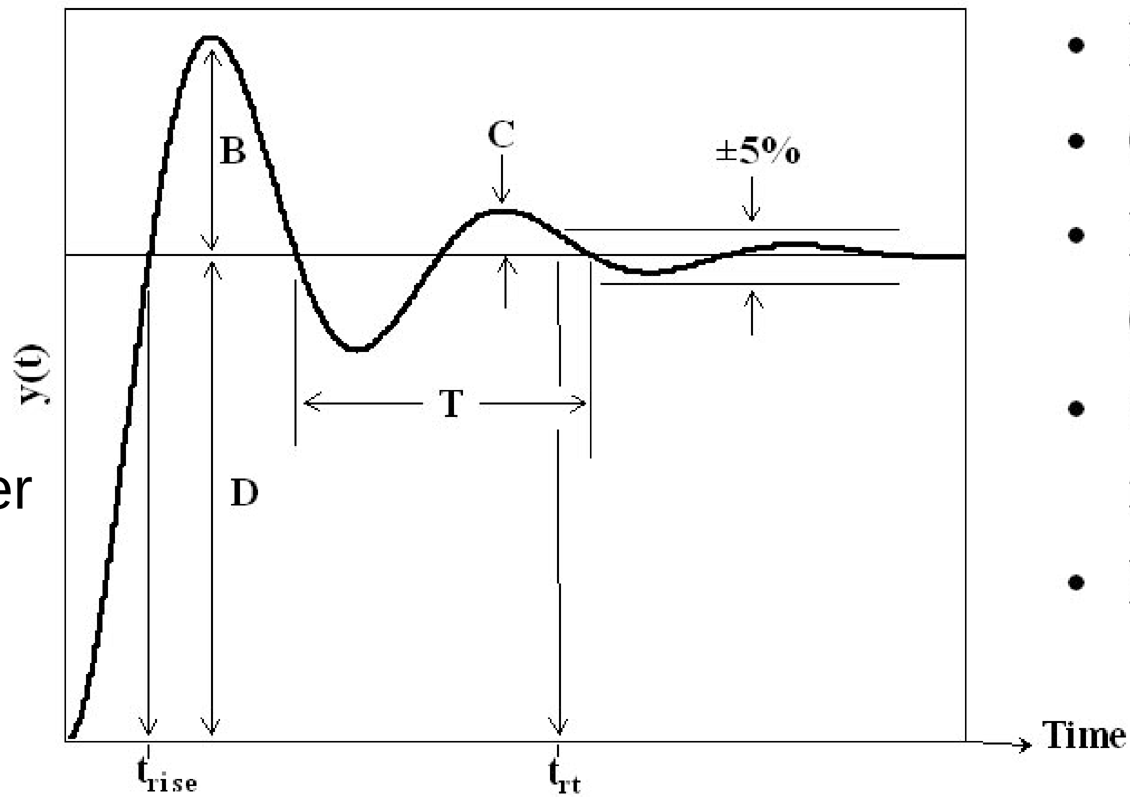
$$s^3 + 10s^2 + 24s + K = 0 \;\Big|\; s = j\omega$$

$$j\omega(-\omega^2 + 24) + K - 10\omega^2 = 0$$

$Re=0: \; K = 10\omega^2$

$Im=0 \quad \omega^2 = 24 \Rightarrow K = 240$

State Observer



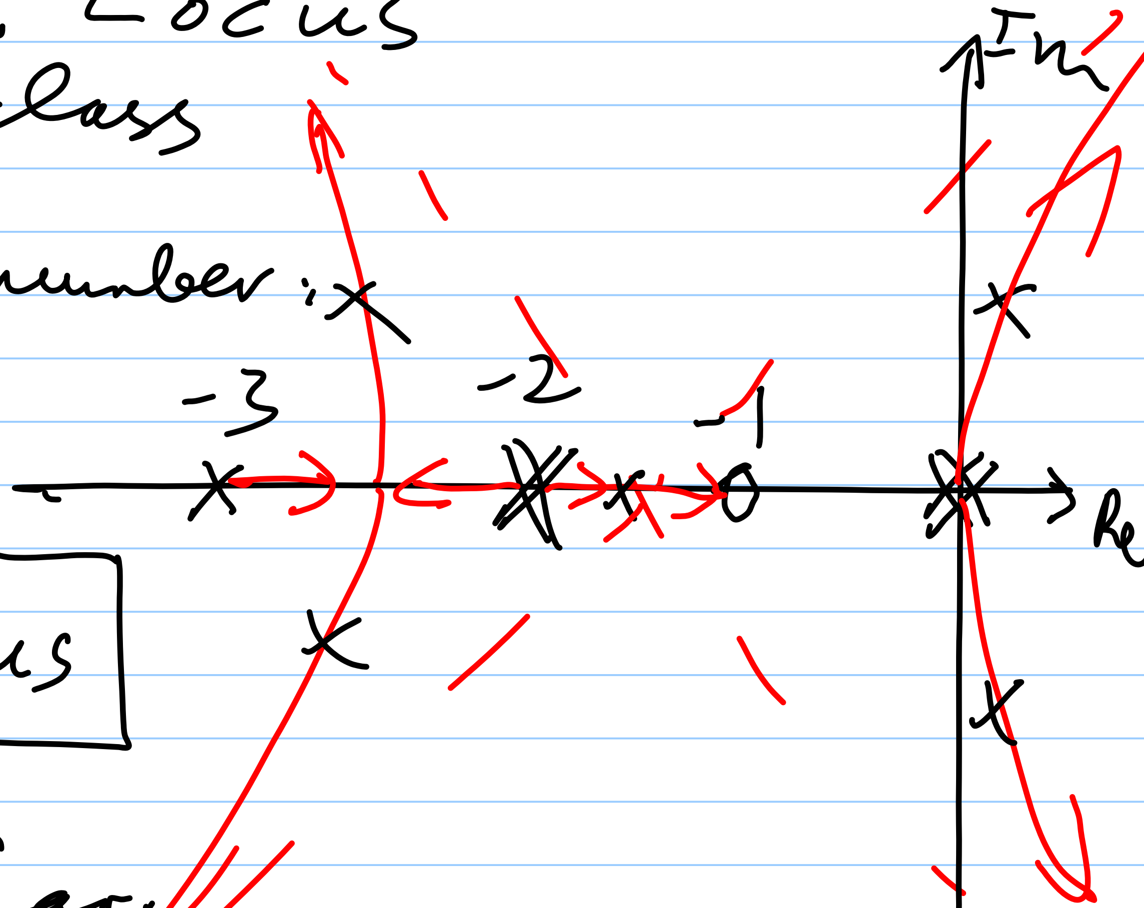
- Rise time
- Overshoot P.O.
- Decay ratio (C/B)
- Settling or response time
- Period (T)

Root Locus

in class

let K = number:

closed loop poles on RLocus

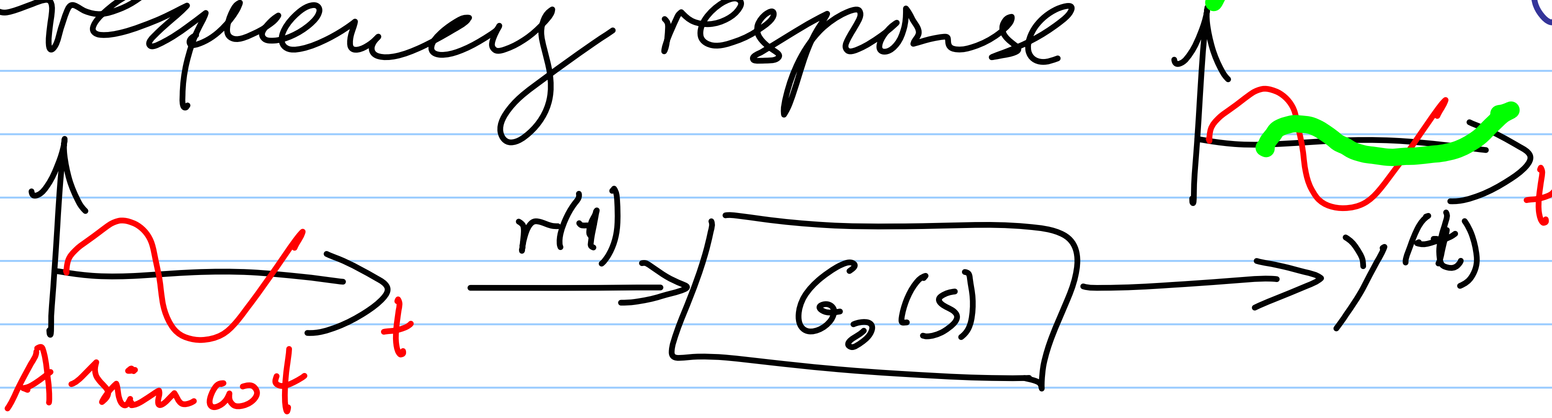


= Solution of the char. equation $1 + K G_0(s) = 0$

$0 < K < \infty$

Frequency response

$y = B\sin(\omega t + \varphi)$



Mathematically: Fourier Transform

$$\frac{Y(j\omega)}{R(j\omega)} = F(j\omega)$$

formally similar to Laplace TF

substitute: $s = j\omega$

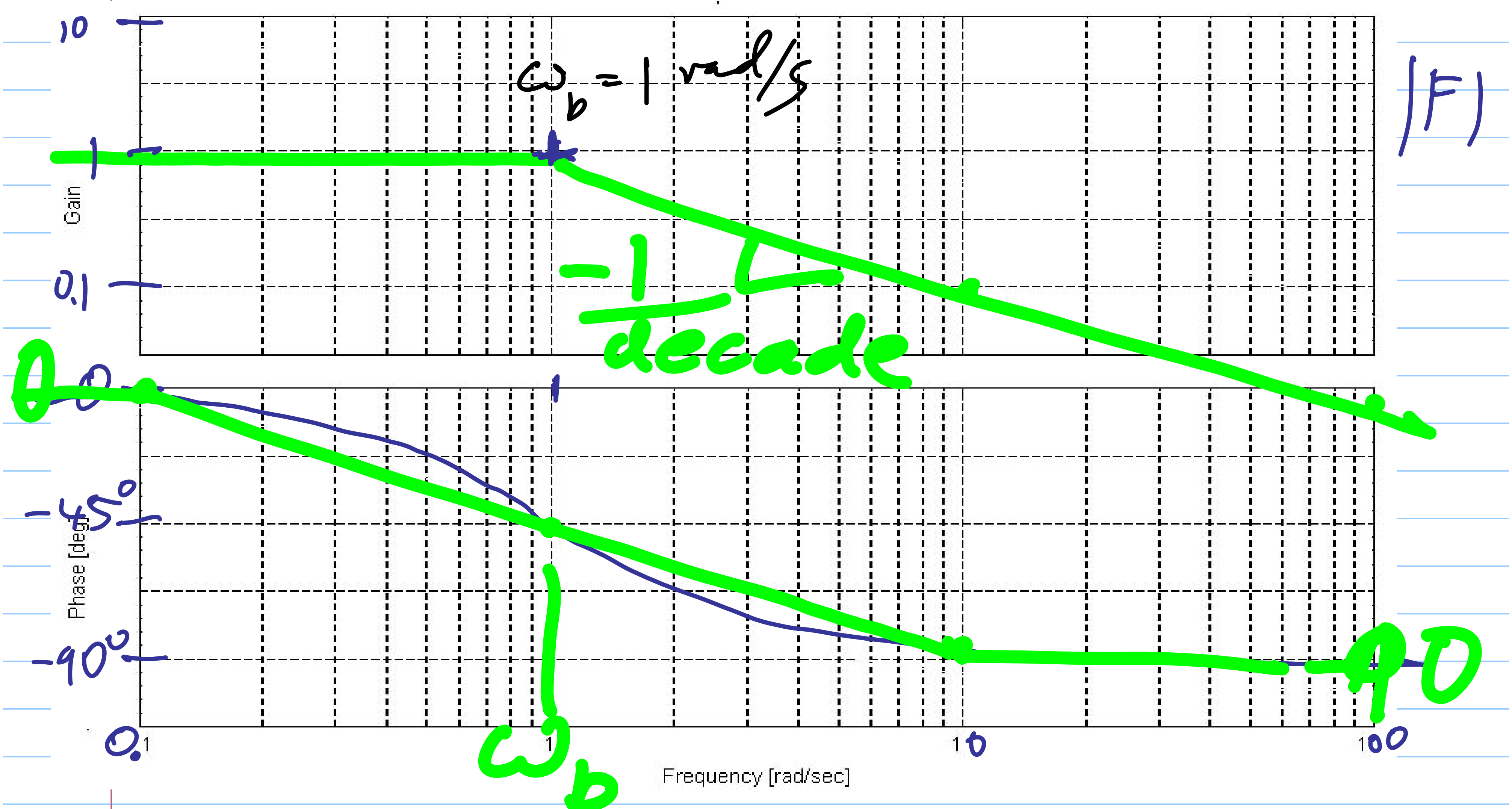
Open-Loop!

Ex.: $T(s) = \dfrac{10}{(s+1)(0.1s+1)}$

$$F(j\omega) = \frac{10}{(j\omega+1)(0.1j\omega+1)}$$

Bode plot of $\dfrac{1}{j\omega+1}$

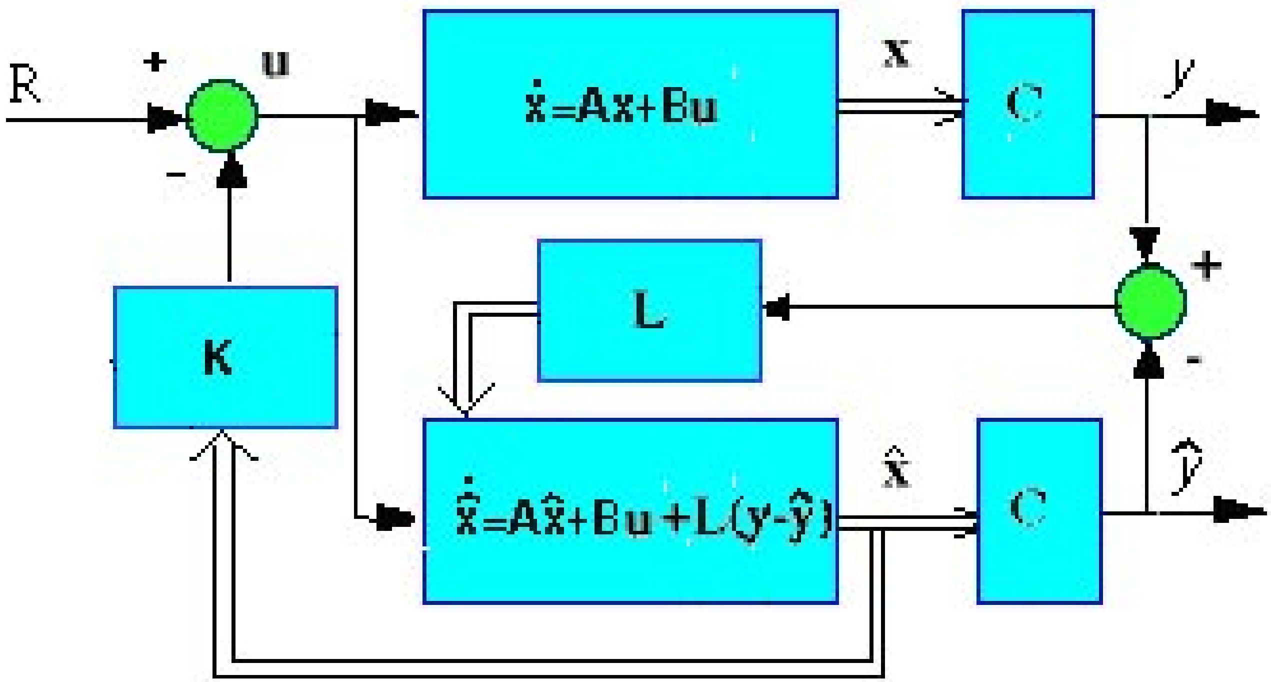
$$\frac{1}{j\omega+1} \qquad \omega_b = \frac{1}{\tau} = 1\,s^{-1} \qquad G_0 = \frac{K}{\tau s+1}$$



for $\omega \gg \omega_b$ $|F| = \dfrac{1}{\sqrt{\omega^2+1}} \approx \dfrac{1}{\omega}$

State Observer

- not all states observed
- design either full state or partial state observer



state DE $\dot{x} = Ax + Bu$ ← insert $u = -Kx$

$y = Cx$

Observer $\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$ estimated state $\hat{x}$

$= (A - LC)\hat{x} + Bu + Ly$

Observer error $e = x - \hat{x}$: $\dot{x} - \dot{\hat{x}} = Ax - A\hat{x} - L(Cx - C\hat{x})$

$= (A - LC)(x - \hat{x})$

$u = -K\hat{x}$

$\dot{x} = Ax - BK\hat{x} = (A - BK)x + BK(x - \hat{x})$

$\dot{e} = (A - LC)e$

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A-BK & BK \\ 0 & A-LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}$$

Observer for 2 states only

$$A = \begin{bmatrix} A_{aa} & A_{ab} \\ A_{ba} & A_{bb} \end{bmatrix}$$

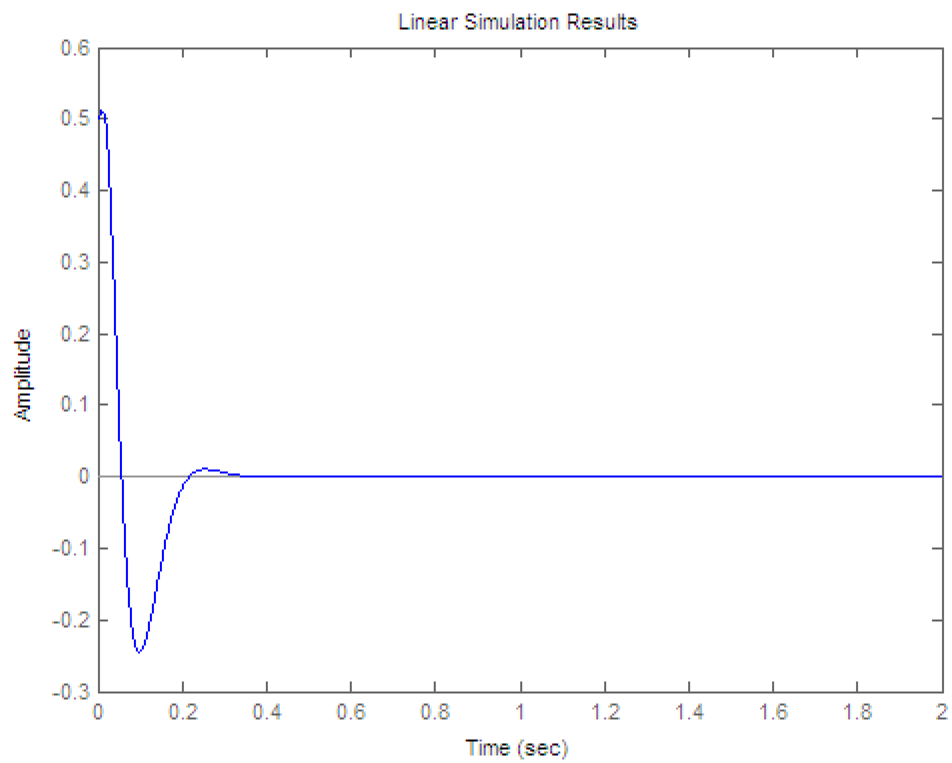
A =

$$\begin{array}{cccc} & & \multicolumn{2}{c}{A_{ab}} \\ 0 & 0 & 1.0000 & 0 \\ 0 & 0 & 0 & 1.0000 \\ -36.0000 & 36.0000 & -0.6000 & 0.6000 \\ 18.0000 & -18.0000 & 0.3000 & -0.3000 \end{array} \quad A_{bb}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ \hline 1 \\ 0 \end{bmatrix} \begin{matrix} B_a \\ \\ B_b \end{matrix}$$

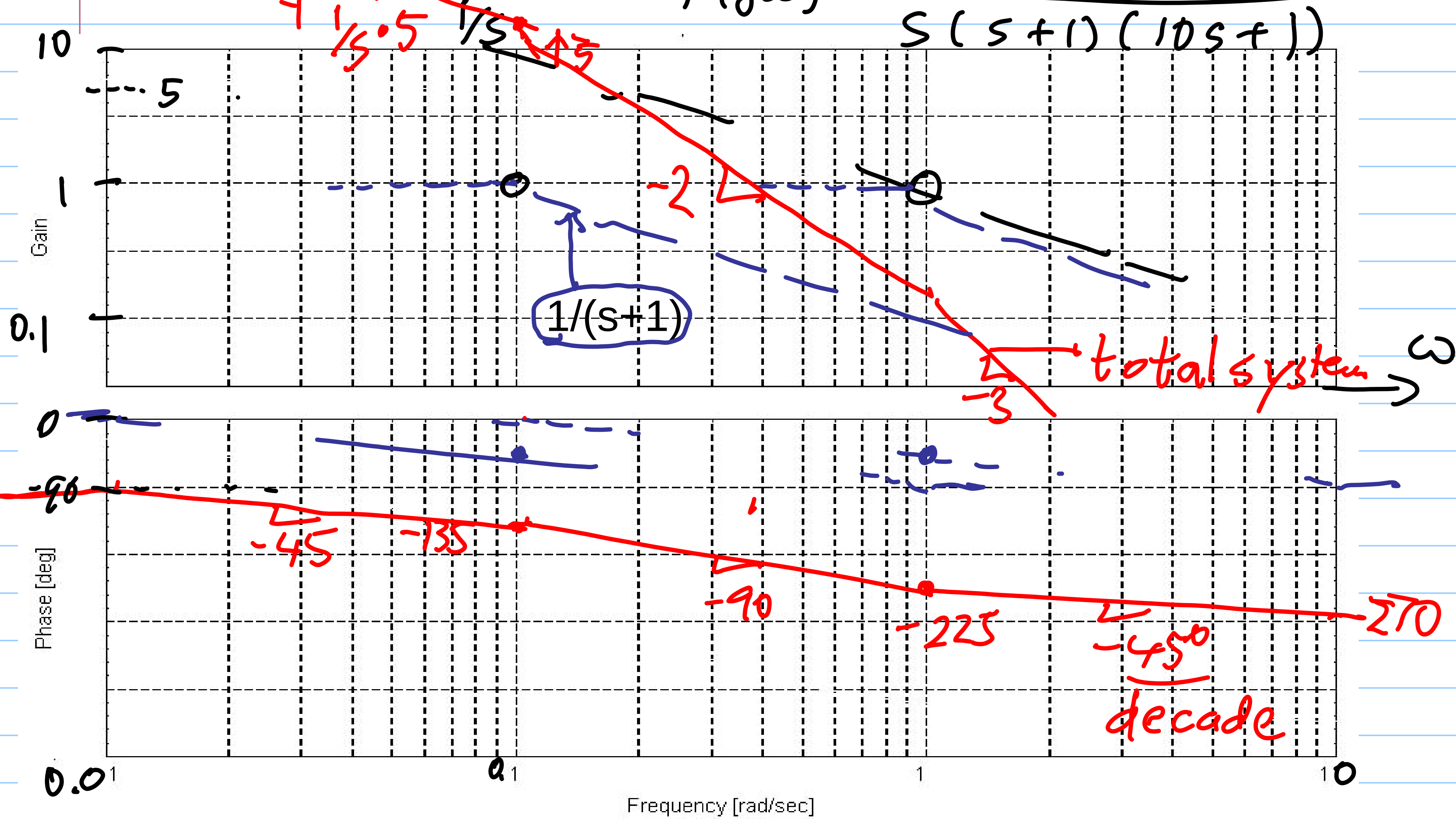
Full State Observer

```
op1 = -100;
op2 = -101;
op3 = -102;
L = place(A',C',[op1 op2 op3])';
At = [A - B*K          B*K
      zeros(size(A))     A - L*C];
Bt = [    B*0.1
       zeros(size(B))];
Ct = [   C        zeros(size(C))];
lsim(At,Bt,Ct,0,zeros(size(t)),t,[x0 x0])
```



Bode Methods easy addition $\omega_{b_1} = 1; \omega_{b2} = 0.1$

$$F(j\omega) = \frac{5}{s(s+1)(10s+1)}$$



$$G = \frac{5}{s(s+1)(0.1s+1)}$$

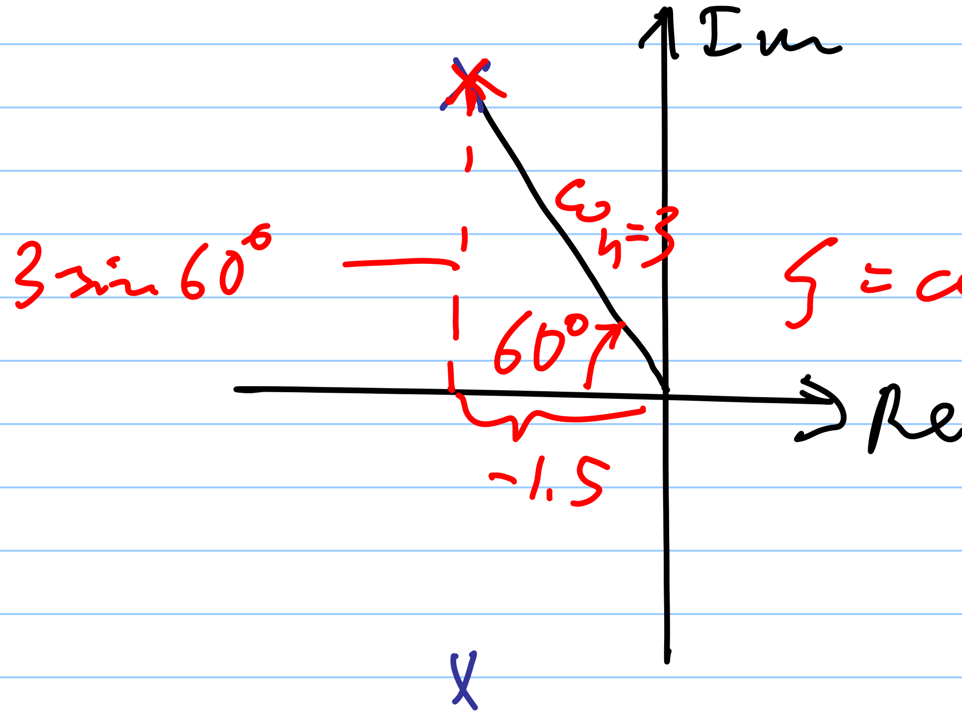
for any ω

ω	$\frac{5}{s}$ $\cdot$	$\frac{1}{s+1}$ $\cdot$	$\frac{1}{0.1s+1}$
1	$\lg 5 - \lg 1$	$-\lg\left\|\sqrt{1^2+1^2}\right\|$	$-\lg\left\|\sqrt{0.1^2+1}\right\|$
$\varphi(1)$	-90	$-\tan^{-1}\frac{1}{1}$	$-\tan^{-1}\frac{0.1}{1}$

compute the $|F|$ and $\varphi(\omega)$ by inserting ω into the respective equation terms

Recall that division is a subtraction in log terms.

Problem 2.4: Pole locations



$$\zeta = \cos^{-1}\varphi = 0.5$$

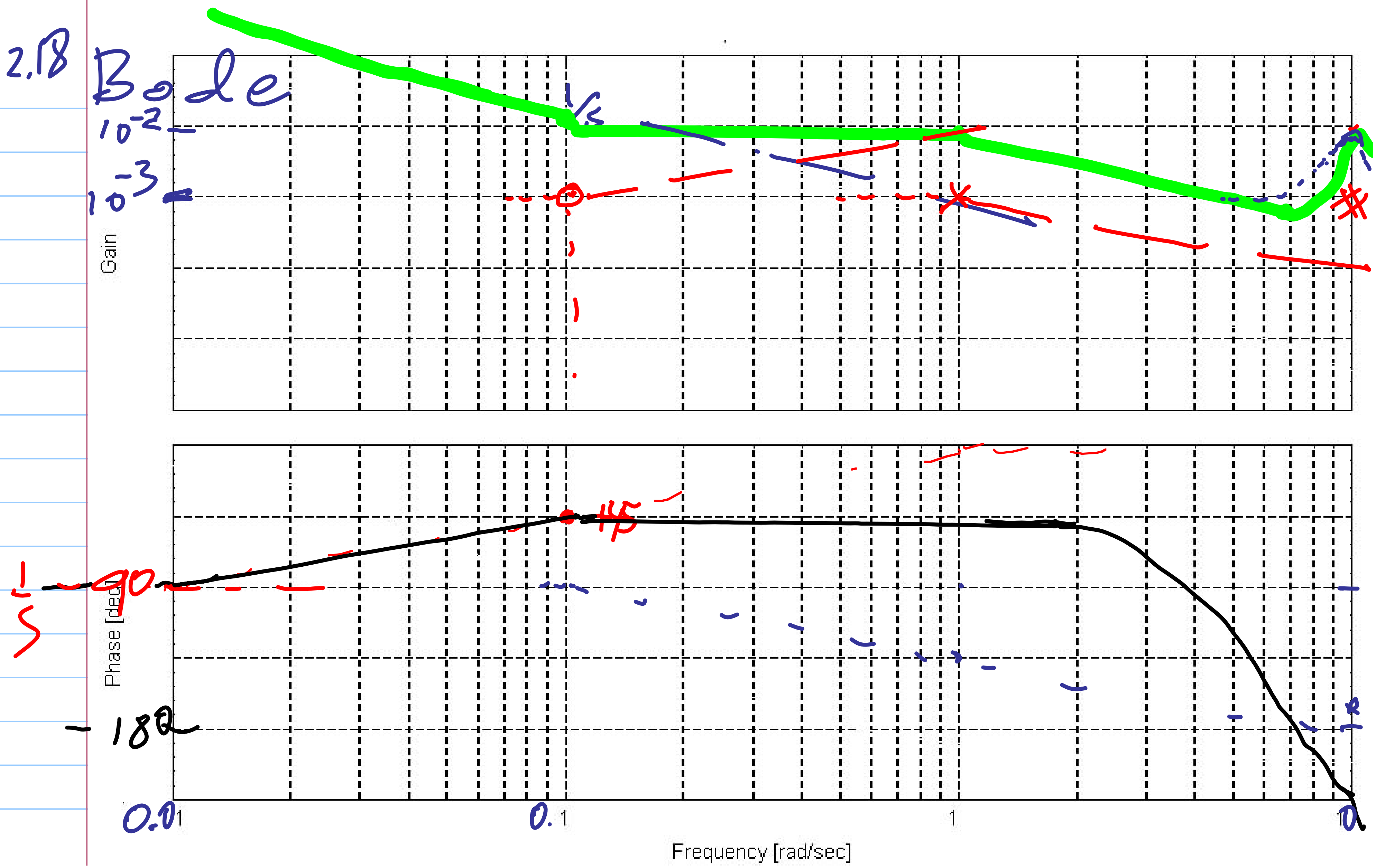
$$\dot{x} = Ax + Bu$$
$$y = Cx$$

SISO

Transition Matrix $\Phi(s) = (sI - A)^{-1}$

$$Y(s) = C' (sI - A)^{-1} B \cdot U(s)$$

2.18 Bode



1/s
10^-2
10^-3
Gain
1/s
-90
45
Phase [deg]
-180
0.01
0.1
1
10
Frequency [rad/sec]

$$F(j\omega) = \frac{100}{s^2 + 2s + 100}$$

$\omega_n = 10$

$\text{poles} = -1 \pm \sqrt{-99}$

$\approx -1 \pm 10 \cdot j$

$$|F| = \frac{100}{\sqrt{(100-\omega^2)^2 + (2j\omega)^2}}$$

let $\omega = 10$;

$$|F(10)| = \frac{100}{\underbrace{\sqrt{(2\omega)^2}}_{\text{small}}}$$

Resonance

$$F(j\omega) = \frac{A}{j\omega + 1 - j\omega_n} + \frac{B}{j\omega + 1 + j \cdot \omega_n}$$

$$\varphi(\omega) = -\tan^{-1}\frac{\omega-\omega_n}{1} - \tan^{-1}\frac{\omega+\omega_n}{1}$$

$$x \text{ in dB} = 20 \lg x$$

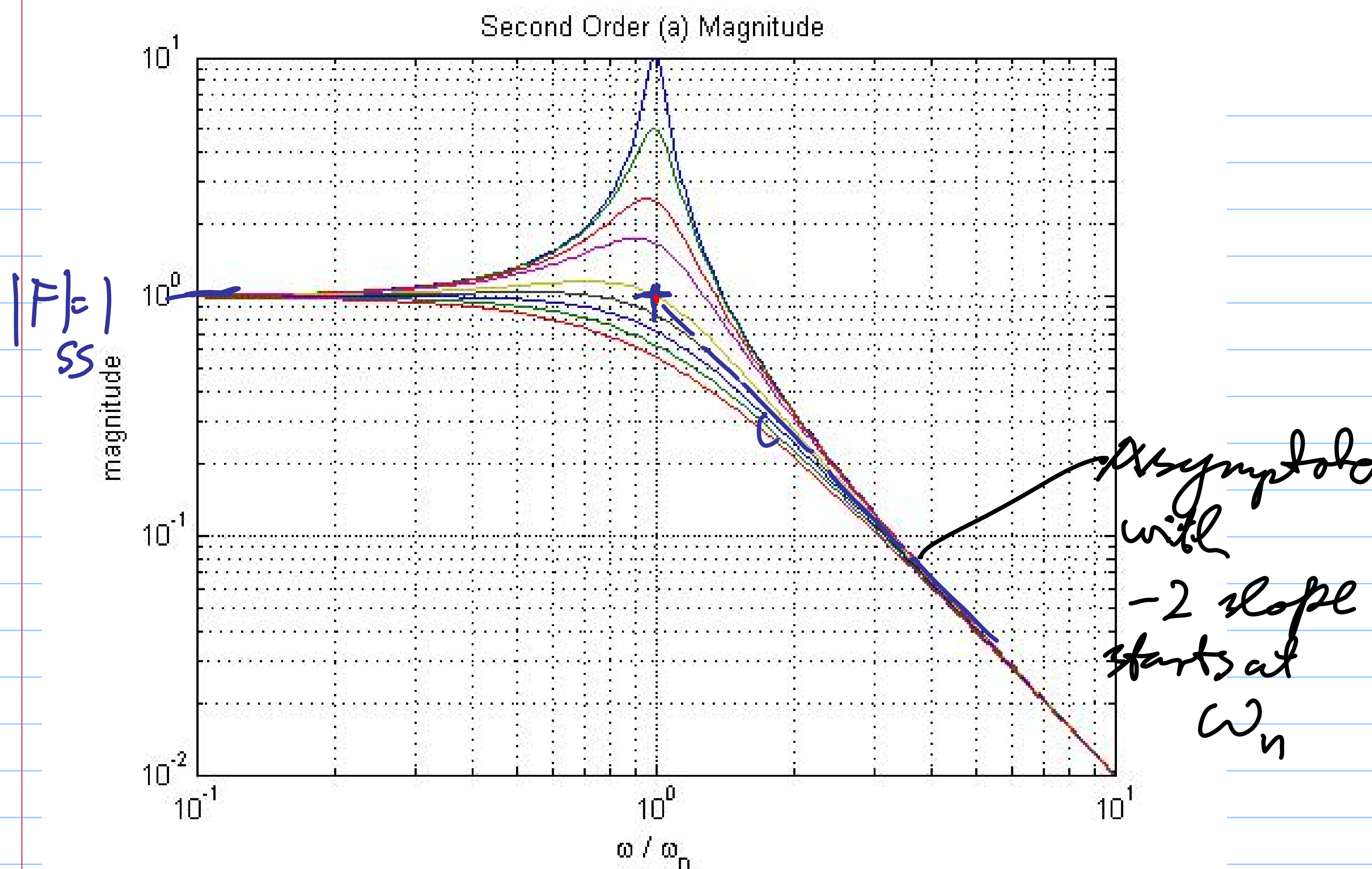
x	x dB
0.1	-20

$$T(s) = \frac{Y(s)}{R(s)} = \frac{s+0.1}{s(s+1)(s^2+2s+100)}$$

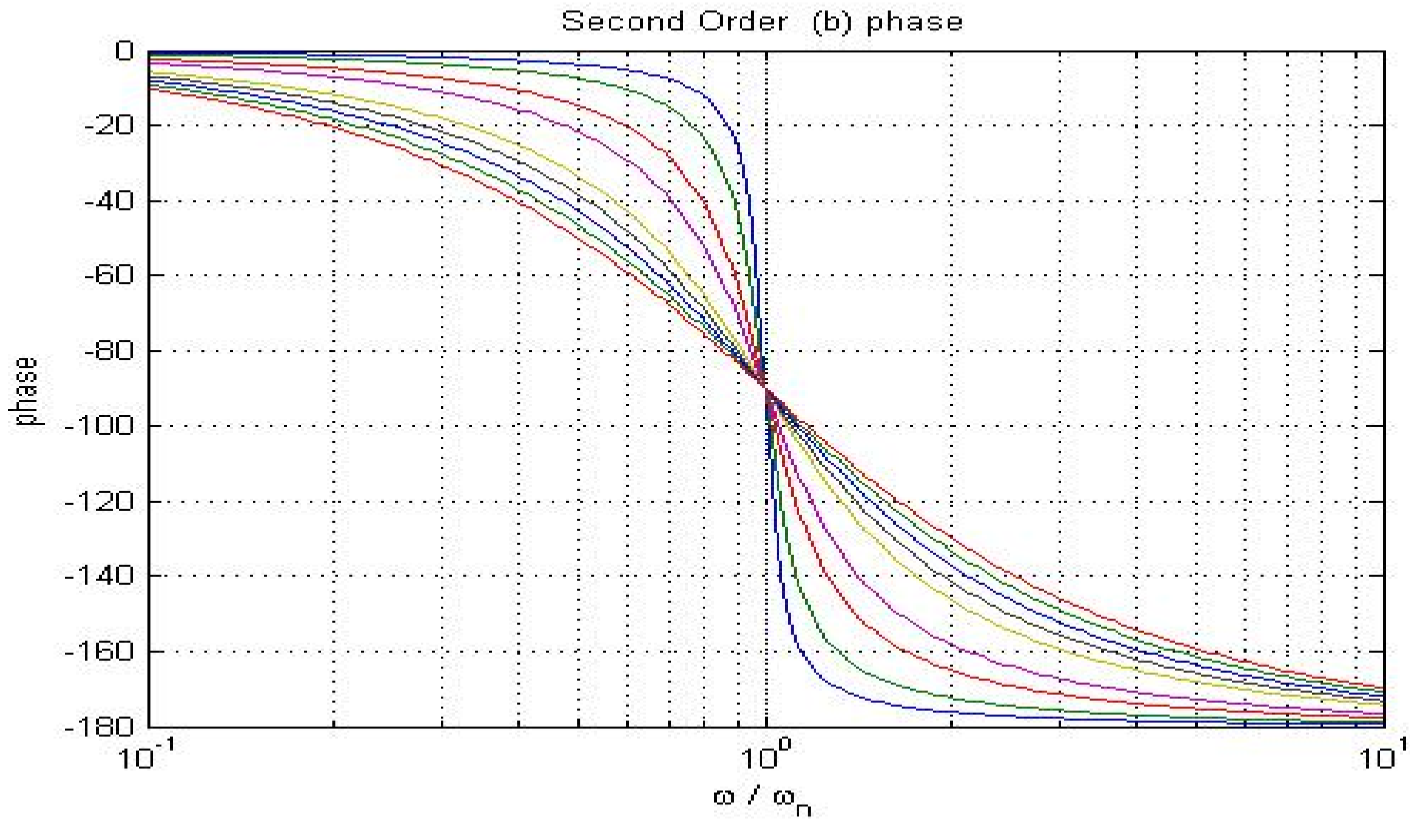
exclu
integr. $K_{ss} = \lim_{s \to 0} T(s) = \frac{0.1}{100}$

$$y = K_{ss} \cdot r$$

First Midterm: Review Mat.



Second Order (a) Magnitude
magnitude
ω / ωn
10^1
10^0
10^-1
10^-2
|F| = 1
ss
Asymptote with -2 slope starts at ωn



Second Order (b) phase
phase
0
-20
-40
-60
-80
-100
-120
-140
-160
-180
10^{-1}
10^{0}
10^{1}
ω / ω_n

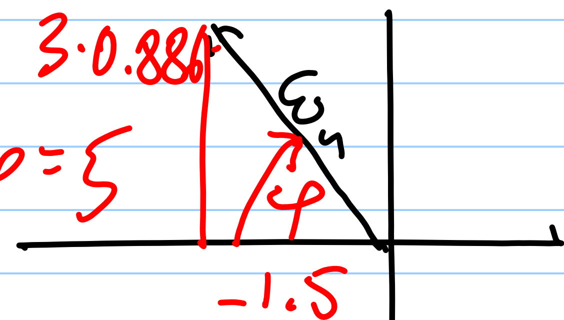
```
%Problem 2.4 DigContr Franklin
A = [  0  1
    0  -4];
B = [0
     1];
C = [1   0  ];
poles = eig(A)
t = 0:0.01:10;
u = 0.1*t;
x0 = [0.5 0 ];
[y,x] = lsim(A,B,C,0,u,t);
h = x(:,2);   %Delta-h is the output of interest
plot(t,h)
pause
p1 = -1.5+ 3*0.887i;
p2 = -1.5- 3*0.887i;

K = place(A,B,[3*p1 3*p2 ]);
[y,x] = lsim(A-B*K,B,C,0,u,t,x0);
%[y,x] = lsim(A,B,C,0,u,t,x0);
h = x(:,1);   %Delta-h is the output of interest
plot(t,h)
pause

%Observer
op1 = 2*p1;
op2 = 2*p2;
L = place(A',C',[3*op1 3*op2 ])';
At = [A - B*K           B*K
     zeros(size(A))     A - L*C];
Bt = [    B*0.1
      zeros(size(B))];
Ct = [   C        zeros(size(C))];
[y2,x]=lsim(At,Bt,Ct,0,zeros(size(t)),t,[x0 x0]);
plot(t,y2)
```

Matlab script for solution of Problem 2.4 (state control with observer)

$\cos^{-1}\varphi = \xi$



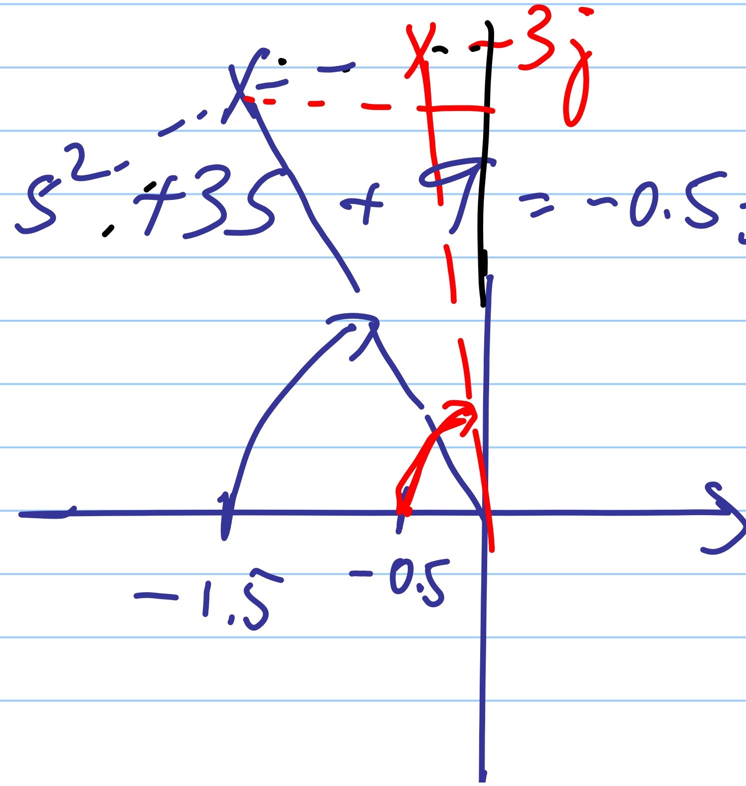
for $\xi = 0.5 \Rightarrow \varphi = 60^0$

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \qquad 0<\zeta<1$$

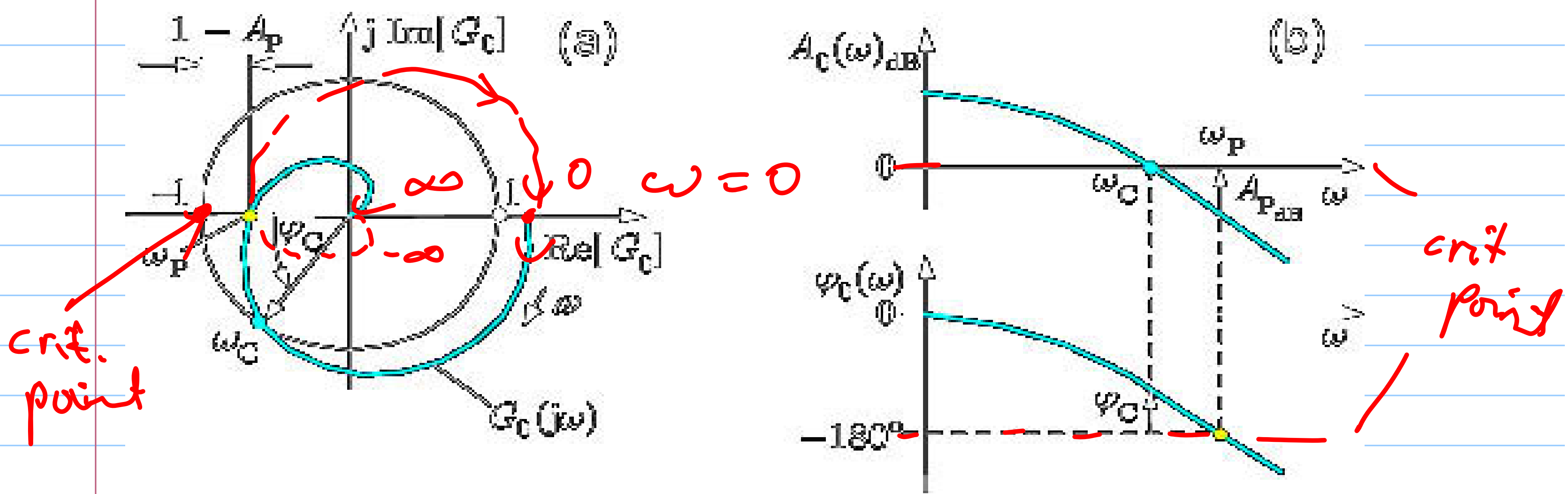
$$p_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

$$s^2 + 3s + 9$$

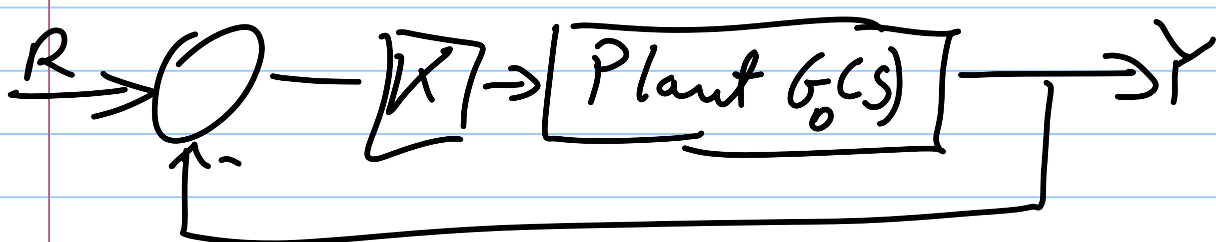
$$(s+3)^2 = s^2 + 3s + 9 = -0.5 \pm j\sqrt{8.75}$$



Nyquist Criterion in Bode plot



Root locus : explicit solution of char. eq.



$$\frac{Y}{R} = \frac{K \cdot G_0(s)}{1 + K \cdot G_0(s)} \; ; \quad \text{char. eq.: } Den(s) = 0$$

for $0 < K < \infty$ $\qquad 1 + \underline{\underline{K}}\, G_0(s) = 0$

Nyquist : open-loop system $K \cdot F_{open}(j\omega)$

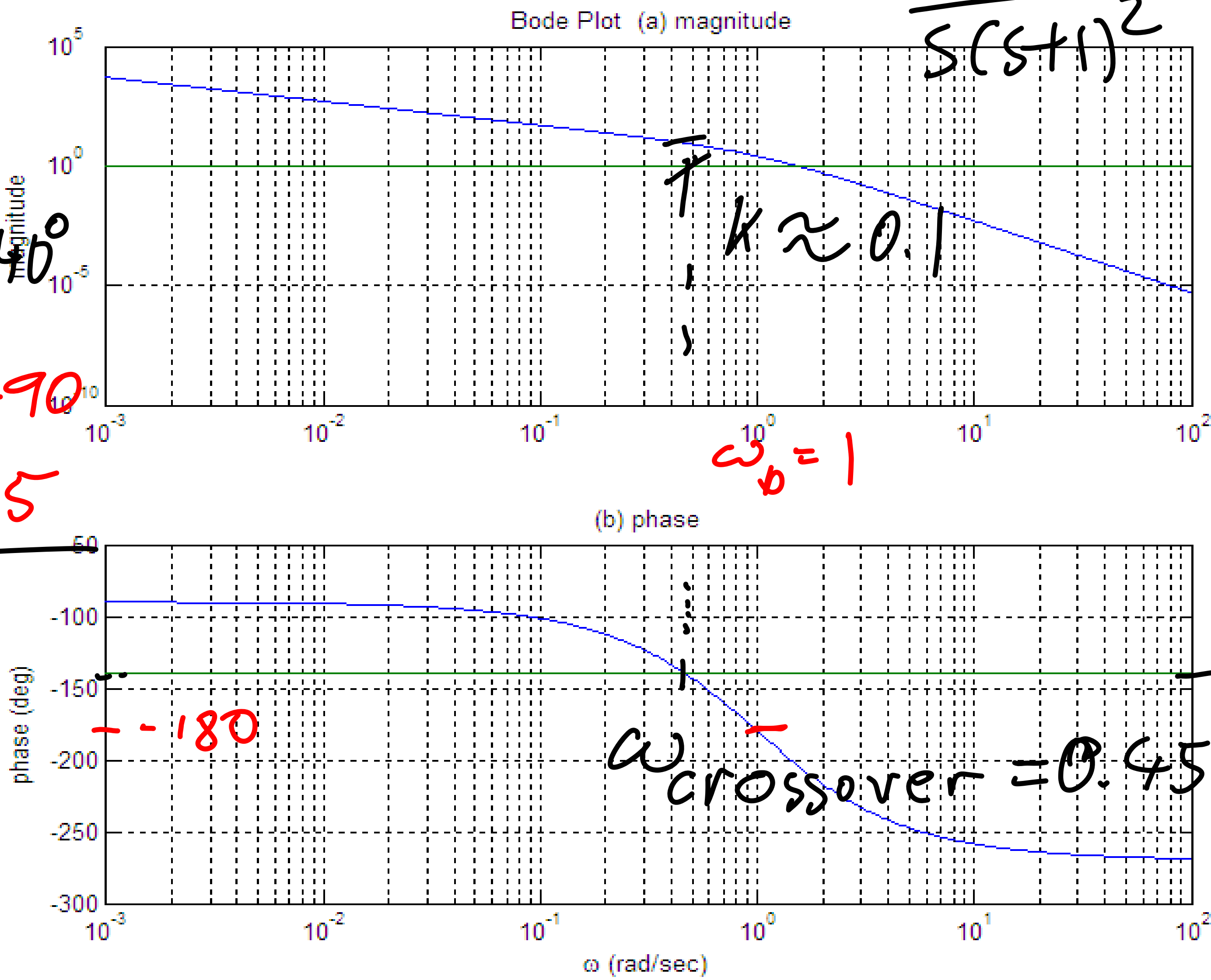
rewrite : $K \cdot F_{open}(j\omega) = -1$

$$\frac{5 \cdot K}{s(s+1)^2}$$

Design for $\varphi_{margin} = 40^\circ$

$\varphi(\omega_b) = -90$
$-45 - 45$

must modify K



$K \approx 0.1$

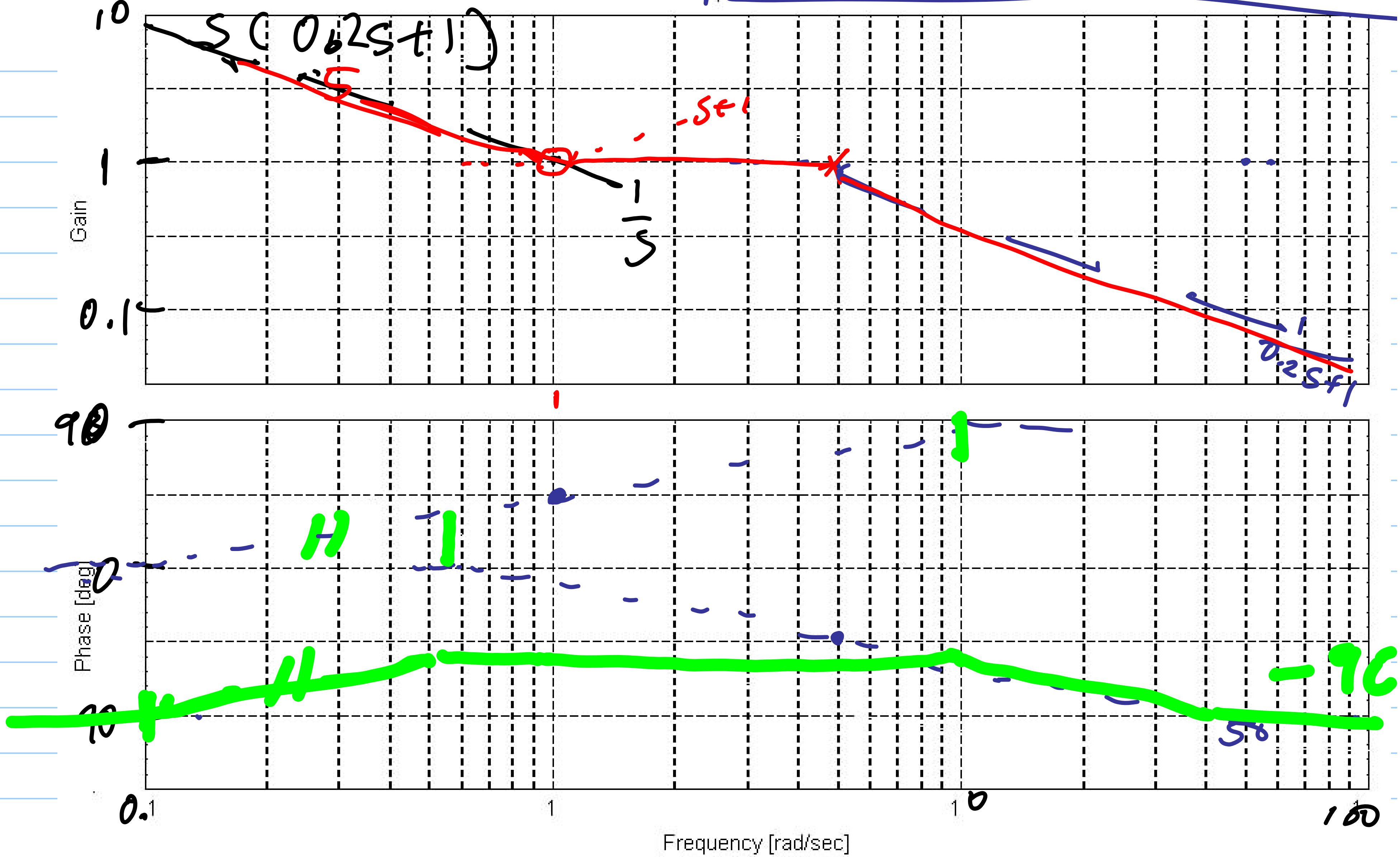
$\omega_b = 1$

-180

$\omega_{crossover} = 0.45 \frac{rad}{s}$

Feb. 26, 2008 Homework

$$\frac{s+1}{s(0.2s+1)}$$



Nyquist in Polar plot