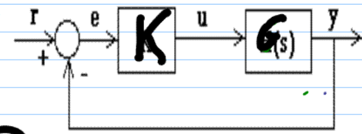
# Chapter 5 Root Locus

Note Title 10/7/2011

The root locus of an (open-loop) transfer function G(s) is a plot of the locations (locus) of all possible closed loop poles with proportional gain k and unity feedback:

Variable $G(s)$




The closed-loop transfer function is:

$0 < K < \infty$

$$\frac{Y(s)}{R(s)} = \frac{KG(s)}{1+KG(s)}$$

and thus the poles of the closed loop system are values

RLocus is plot of all roots of the CLOSED LOOP char. eq. for $0 < K < \infty$



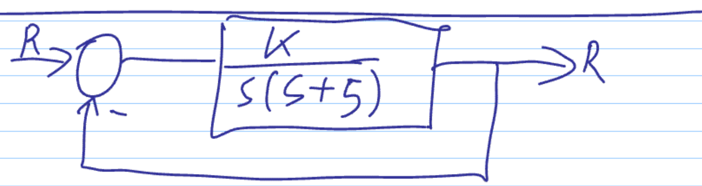


for $K = 0$ RL starts in open-loop poles and terminates in $\infty$ or in open-loop zeros.

$$\frac{K}{s(0.5s+1)(s+1)}$$

$0 \quad -2 \quad -1$

Ex.:



$$\frac{Y}{R} = \frac{K}{s(s+5)+K}$$

$s(s+5)+K = 0 \qquad s^2+5s+K=0$

$$p_{1,2} = -\frac{5}{2} \pm \sqrt{\left(\frac{5}{2}\right)^2 - K}$$

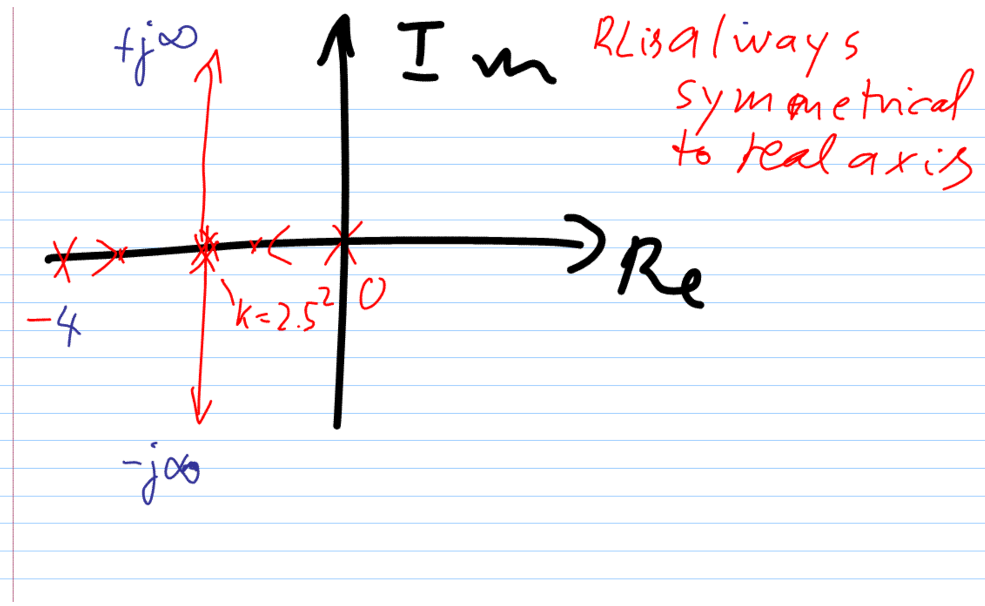
$2.5^2 = 6.25$

for $K > 6.25$: $Re(p_i) = -2.5$

$K \to \infty \qquad Im(p_i) \Rightarrow \pm j\infty$



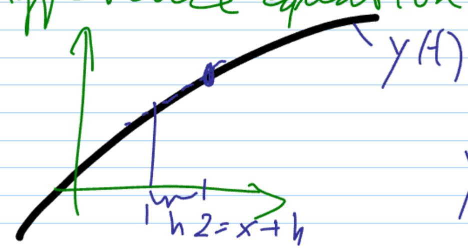




Controller output — $U(s) = \dfrac{K(s+z_1)(s+z_2)}{(s+p_1)(s+p_2)} \cdot E(s)$

$a\ddot{u} + b\dot{u} + c \cdot u = \alpha \ddot{e} + \beta \dot{e} + \gamma \cdot e(t)$

Difference equation




$\dot{y} = f(x)$

start at 1

$y_2 = y_1 + h \cdot \left(\dfrac{df}{dt}\right)_1$



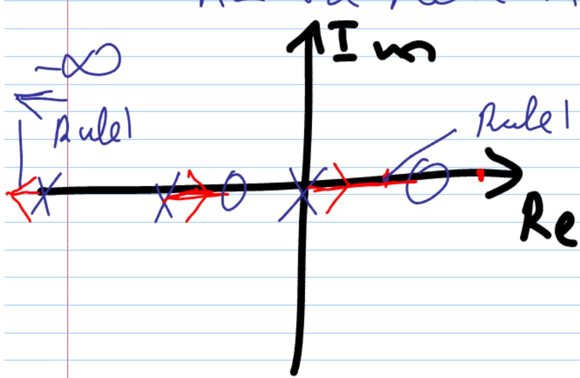


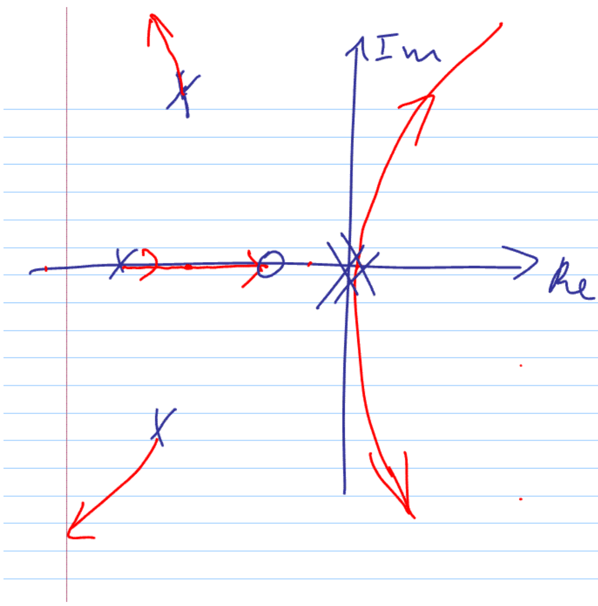
Details see ch. 8

Digital Control

Rule 1:

RL on Real Axis






$$s^2+10s+K=0$$

$$s_{1,2}=-5\pm\sqrt{25-K}$$

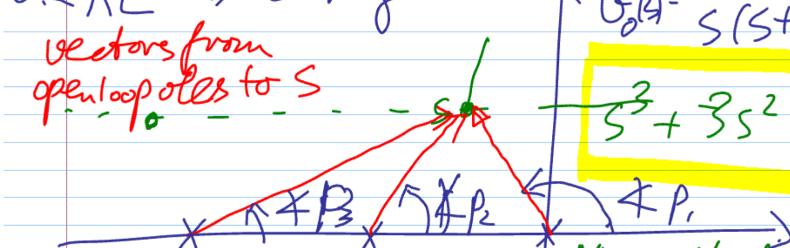




## 5.2 RL Theory

Trial Point

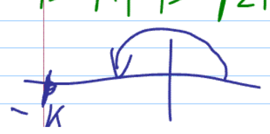
$$G_o(s)=\frac{K}{s(s+1)(s+2)}$$




$$\boxed{s^3+3s^2+2s+K=0}$$

$s = Re + j\,Im$

$$|s-p_1|\cdot|s-p_2|\cdot|s-p_3| = K$$




Magnitude condition

$$|s^3+3s^2+2s| = |-K|$$

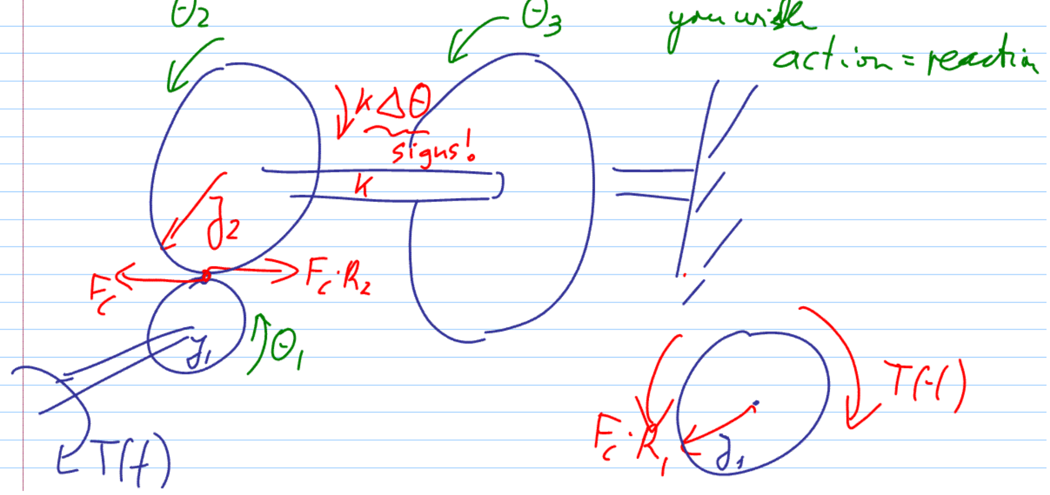
Angle condition

$$\angle p_1+\angle p_2+\angle p_3=\angle -K = -180^\circ$$

Model

1. Frame(s)
2. as many FBD's as you wish

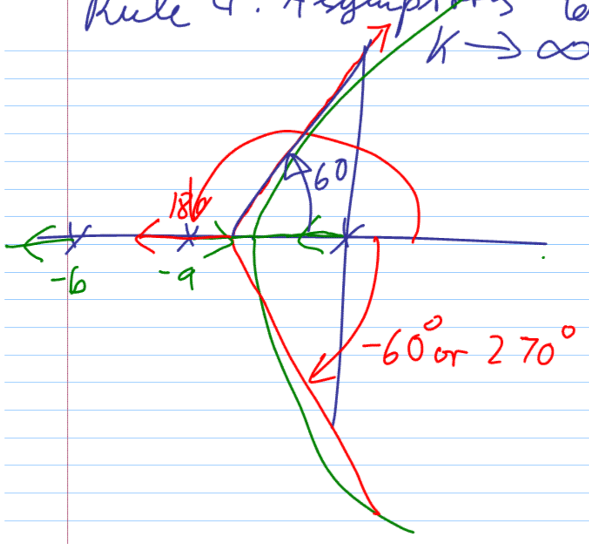
action = reaction








Rule 4: Asymptotes ? let $G(s) = \dfrac{K}{s(s+a)(s+b)}$

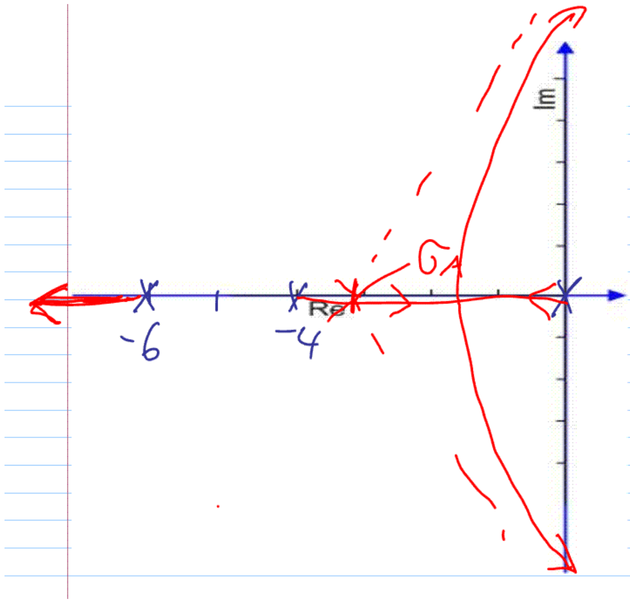



$$\sum \measuredangle \varphi_i = 180^\circ$$

$$3 \cdot \measuredangle \varphi_i = 180$$

$$\varphi_i = \frac{180 \pm k \cdot 360^\circ}{3}$$

$$k = 0, 1, 2$$




```
np=10;
dp= [1 10 24 0];
sysol=tf(np,dp);
```

$\#p - \#z = 3$

$$G_0(s) = \frac{10}{s^3+10s^2+24s}$$

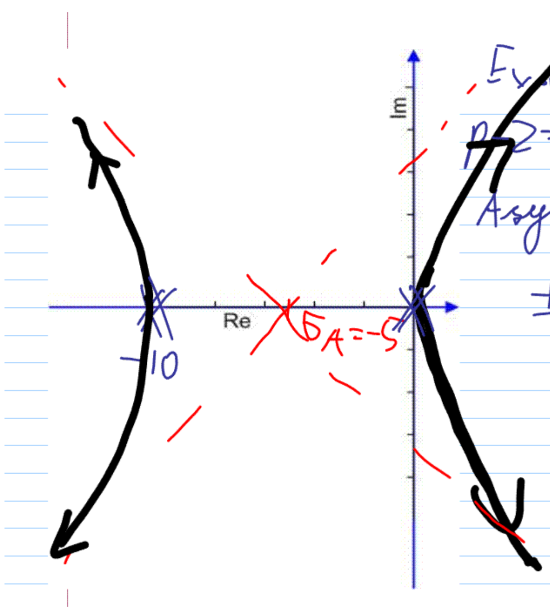
Open-loop poles at 0, −4, −6

$$\sigma_A = \frac{(0-4-6)-0}{3}$$

$$\boxed{\sigma_A = -\frac{10}{3}}$$







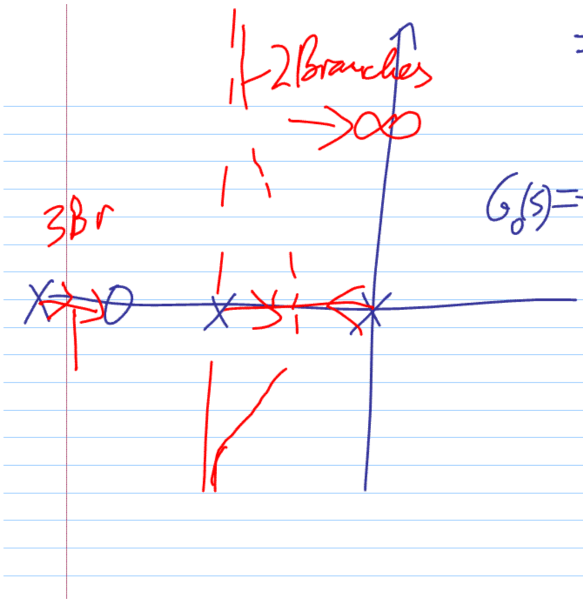

Ex: $G_0(s) = \dfrac{K}{s^2(s+10)^2}$

$n_p = 4$

Asymptotes: $\phi_c = \dfrac{180 + k\cdot 360}{4}$

$\pm 45^\circ \; \pm 135^\circ$

$$\sigma_A = \frac{(-0-0-10-10)-0}{4} = -5$$

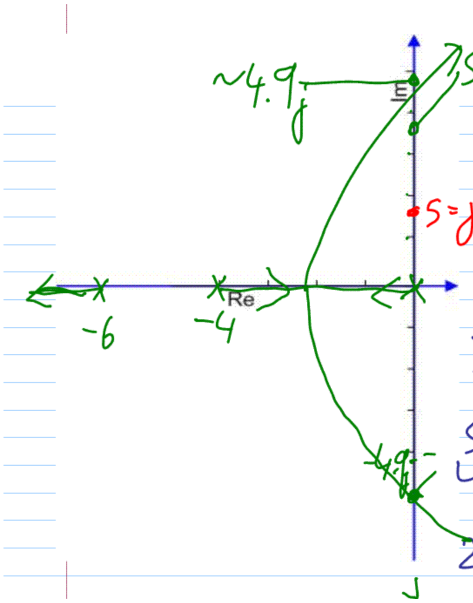



\# of Branches of RL = order of char. eq.

$$G_o(s) = \frac{K(s+3)}{s(s+2)(s+4)}$$








$s = j\omega + 0$

$$G_o(s) = \frac{10K}{s^3+10s^2+24s}$$

Num + Den = 0 $\qquad s^2 = -\omega^2$

$$s^3 + 10s^2 + 24s + 10K = 0 \quad \text{Char. eq.}$$

Im + Re = 0

$$\underbrace{s(s^2+24)}_{Im=0} + \underbrace{10s^2 + 10K}_{Re=0} = 0$$

$$24 - \omega^2 = 0 \quad \| \quad -10\omega^2 + 10K = 0$$

Group polynomial: odd $\rightarrow Im(s) = 0$

even $Re(s) = 0$

$\omega^2 = 24 \Rightarrow \omega = \pm\sqrt{24} \cdot j$

insert into $Re = 0$: $-10 \cdot 24 + 10K = 0$

$K_{crit} = 24$





Project

Time management

Estimation of Effort required

Modeling: ME 330

Why? project is open-ended

prep. for reality

Steady state gain (Ch.3)

The steady state gain of a system is the ratio **output/input** in the steady state. The steady state gain is Yss/Rss = G(s = 0)
Caution: Disregard Integrators!

$$\frac{Y}{R} = \frac{K(s+z_1)\cdots}{(s+p_1)(s+p_2)\cdots} \qquad K_{ss} = \frac{K \cdot z_1 \cdots}{p_1 \cdot p_2 \cdots}$$

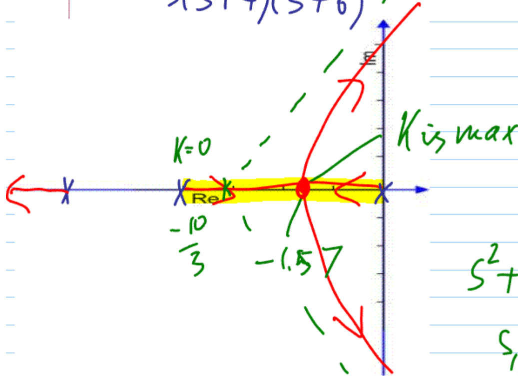




**Rule #5 (Breakaway Points):** For $K > 0$, the root locus breaks away from the real axis at points of relative maximum $K$ and re-enters the real axis at points of relative minimum $K$. i.e., breakaway and re-entry occur at points $sB$ where
dK/ds = 0 *at* $_{s=sB}$

$$G_o = \frac{K}{s(s+4)(s+6)}$$

eval: $\frac{dK}{ds}$




$$G_o + 1 = 0$$

$$s(s+4)(s+6) + K = 0$$

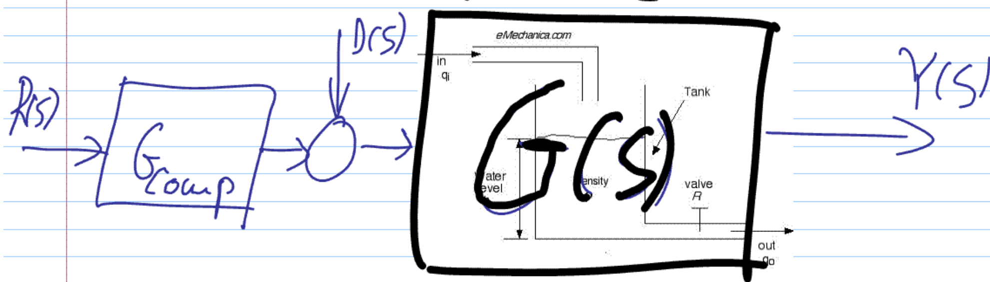
$$K(s) = -[s^3 + 10s^2 + 24s]$$

$$\frac{dK}{ds} = -[3s^2 + 20s + 24]$$

$$s^2 + \frac{20}{3}s + 8 = 0$$

$$s_{12} = -\frac{20}{6} \pm \sqrt{\left(\frac{20}{6}\right)^2 - 8} \qquad s_1 = -1.57$$

poles, zeros

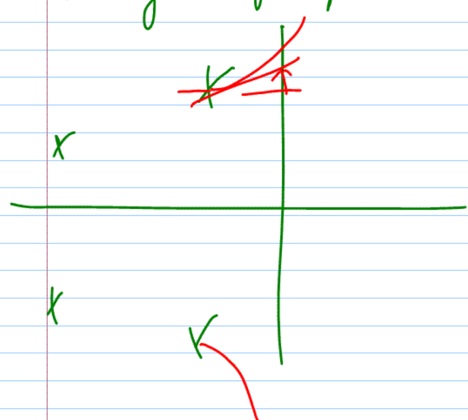



want $Y(s) = 1 \cdot R(s)$ we get $G_p^{-1} \cdot G_p = 1$

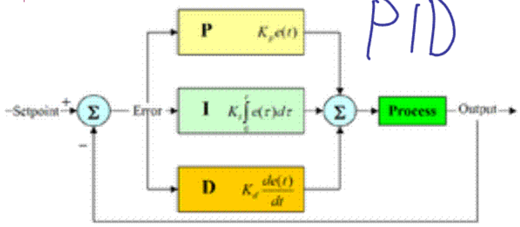




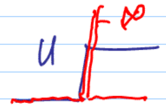
Angles of Departure (read!)



## 5.4 Root Locus Design : Compensators




PID




u(t) = [redacted] $K_p e(t) + K_i \int_0^t e(\tau)\, d\tau + K_d \frac{d}{dt} e(t)$

$$\frac{E}{U} = K_p + \frac{K_i}{s} + K_d s = \frac{K_p \cdot s + K_i + K_d s^2}{s}$$



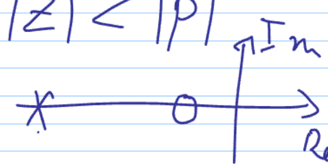


$$= \frac{(T_a s+1)(T_B s+1)}{s}$$ Differentiator

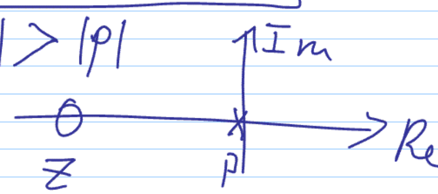
**PD-controller**

more realistic

Lead $= \frac{s+Z}{s+P}$ $\quad |Z| < |P|$



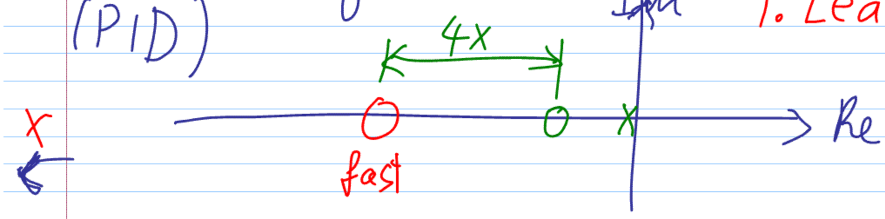

**PI** Lag $= \frac{s+Z}{s+P}$ $\quad |Z| > |P|$




slow, $|P| \approx 0$

Lead-Lag combination (PID)

1. Lead




$$G_{Comp} = K \underbrace{\frac{s+z_{Lead}}{s+p_{Lead}}}_{fast} \cdot \frac{s+z_{Lag}}{s+p_{Lag}}$$



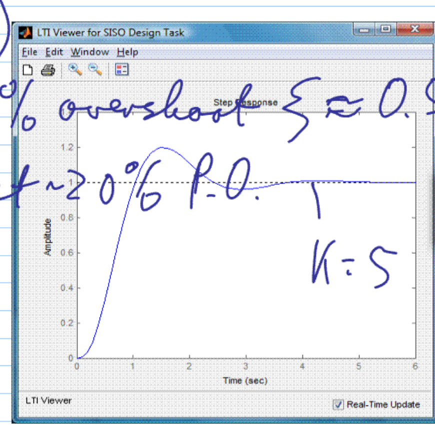


Lead Design Example

Plant $G(s) = \frac{K}{s(s+4)(s+6)}$

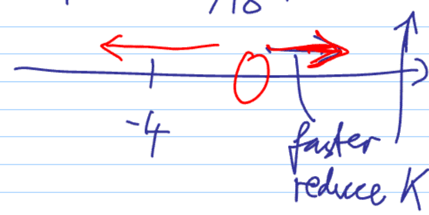
P-control, design for 20% overshoot $\zeta \approx 0.5$

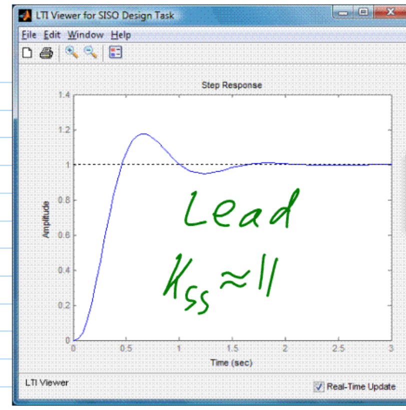
$\boxed{\text{vary } K}$ for $K = 5$ we get ~20% P.O.



Lead

$$G_{lead} \approx 11 \frac{s/3 + 1}{s/18 + 1}$$



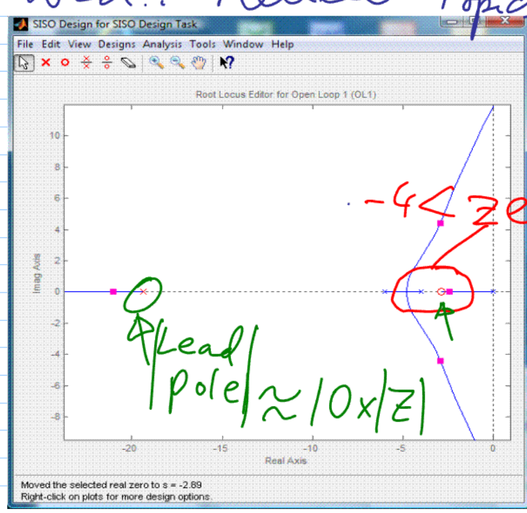

$$\frac{s+3}{s+18} \cdot \frac{18}{3}$$





Wed: Review Topics Ch. 3.5 . . . 5

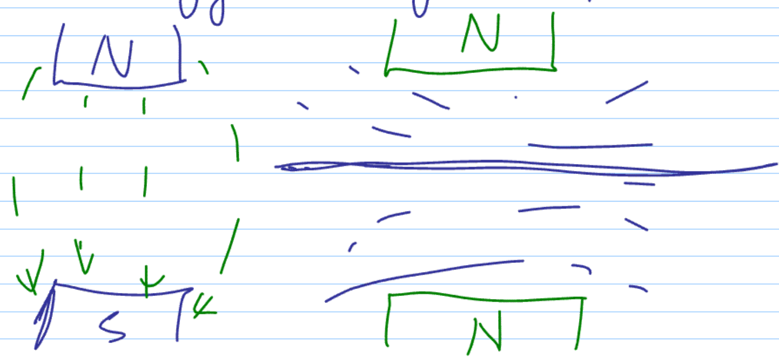
poles and zeros




X→ ←X repel

O attractor

analogy to magnetic fields



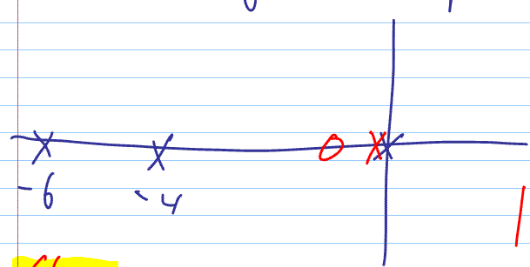





Lead $G_{Lead} = \frac{s+z}{s+p} \approx PD \quad |p| \gg |z|$

Lag $G_{Lag} = \frac{s+z}{s+p} \approx PI \quad |z| \gg |p|$

$G_{Plant} = \frac{1}{s(s+4)(s+6)}$




Place comp pole close to origin

$|z| \approx 10|p|$

Slow

Purpose of Lag: Reduce S.S. error

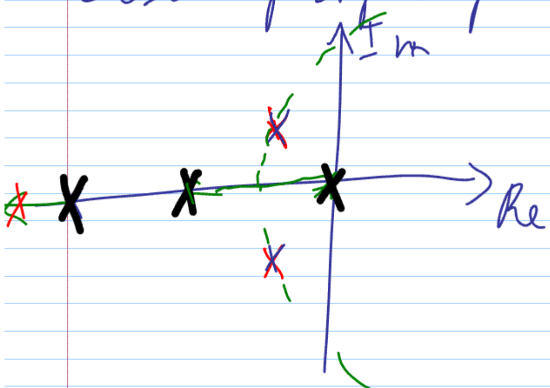
When using combined Lead-Lag

1. Design Lead (fast)
2. Add Lag

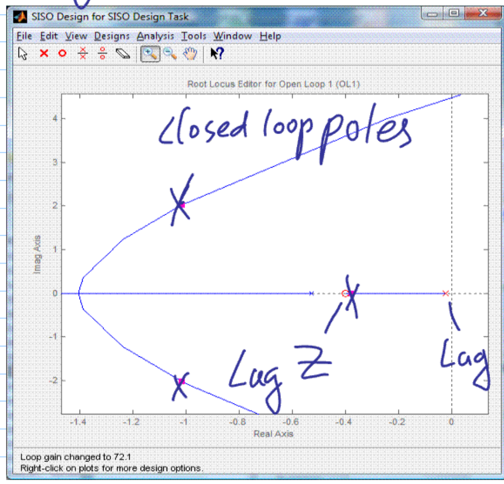




Relation of closed Loop poles in R.L. with closed loop step response. See Ch. 3.4



$$Lag = \frac{s+0.4}{s+0.02} \qquad K_{ss} = 72 \cdot \frac{s/0.4 + 1}{s/0.02 + 1}$$

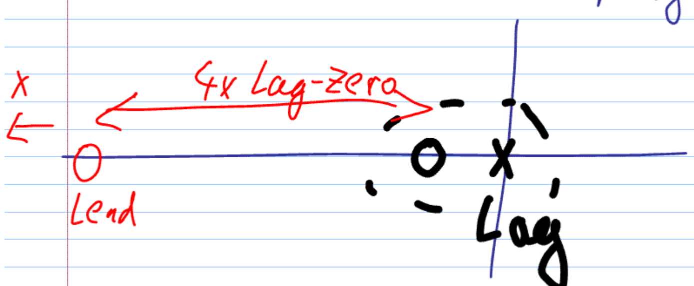







Lead-Lag controller (PID)

$$G_{comp} = \frac{s + z_{lead}}{s + p_{lead}} \cdot \frac{s + z_{lag}}{s + p_{lag}}$$



use only for $\zeta < 1$ will post Notes on web

$$G(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{9}{s^2+4s+9}$$

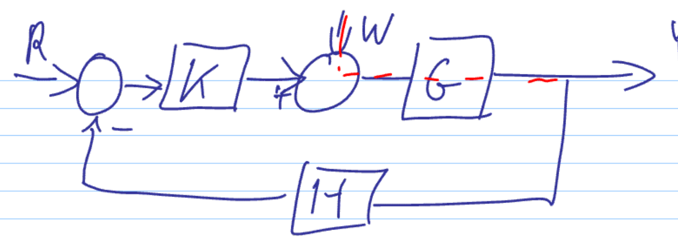
$$\omega_n = 3$$

$$2\zeta\omega_n = 4 \rightarrow \zeta = \frac{4}{2\cdot 3} = \frac{2}{3}$$

when $\zeta \geq 1 \rightarrow$ 2 real poles

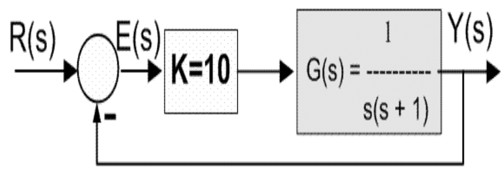







$$Y = G\cdot W + KG(R - H\cdot Y)$$

$$Y = \frac{G\cdot W}{1+KGH} + \frac{KG\cdot}{1+KGH}R$$



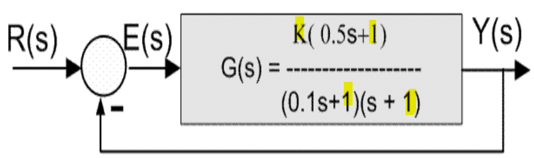

$$E_{ss} = \lim_{s \to 0} s \cdot \frac{1}{s} \frac{1}{s(s+1) + K}$$

$$E = \frac{1}{1+G_0} = \frac{1}{1 + \frac{10}{s(s+1)}} = \frac{1}{1+\infty}$$

$$e_{ss} \to 0$$



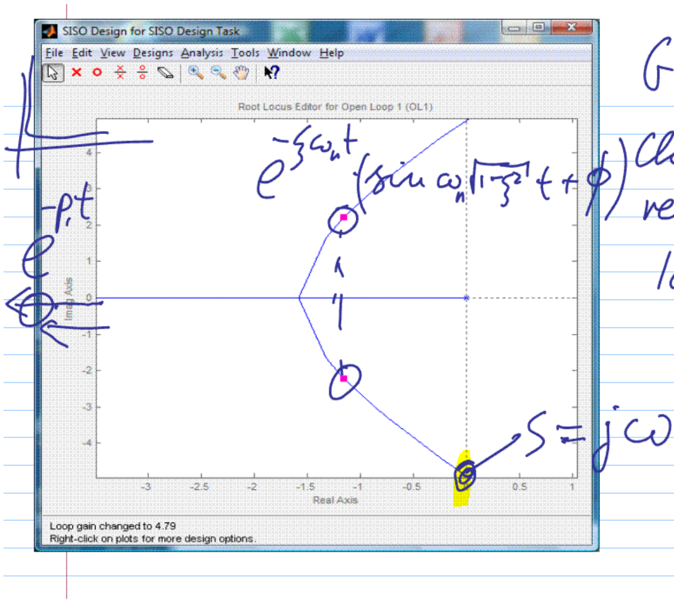





Ramp

$$K_v = \lim_{s \to 0} s \cdot G_0(s) = 0$$

$$e_{ss} = \frac{1}{K_v} \Rightarrow \infty$$




$$G_0 = \frac{10}{s(s+4)(s+6)}$$

Choice $K = 4.78$ results in 3 closed loop poles





Im. axis crossings: find $K_{crit}$, $\omega_n$ at

$s = j\omega + 0 \cdot Re$

$$s^3 + 10s^2 + 24s + 10 \cdot K = 0$$

$Im = 0$: $\quad j\omega(-\omega^2 + 24) = 0 \qquad \omega_n = \sqrt{24}$

$Re = 0$: $\quad 10 \cdot (-\omega^2) + 10K = 0$

$10K - 240 = 0 \qquad K_{crit} = 24$

Add zero $\quad G_0 = \dfrac{10(s+z)}{s(s+4)(s+6)}$

$\# A s = 2 = \# p_i - \# z_i$